Def: Let H be a symmetric $n \times n$ matrix.
Then H is said to be

(1) positive definite if $\vec{x}^T H \vec{x} > 0$
for all column vector $\vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$

(2) negative definite if $\vec{x}^T H \vec{x} < 0$
for all column vector $\vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$

(3) indefinite if $\exists$ column vectors $\vec{x}, \vec{y} \in \mathbb{R}^n \setminus \{\vec{0}\}$
such that $\vec{x}^T H \vec{x} > 0$ and $\vec{y}^T H \vec{y} < 0$

Remark: These are not all possibilities : $\exists$ sym. matrix which is not positive definite, negative definite, nor indefinite.

Then the discussion above implies

Thm (Second Derivative Test)

Let $\begin{cases} \bullet\ f: \Omega \to \mathbb{R} \text{ be } C^2,\ \Omega \subseteq \mathbb{R}^n, \text{ open} \\ \bullet\ \vec{a} \in \Omega \text{ such that } \vec{\nabla} f(\vec{a}) = \vec{0} \end{cases}$

Then

$$Hf(\vec{a}) \text{ is } \begin{cases} \text{positive definite} & \Rightarrow \vec{a} \text{ is a local min} \\ \text{negative definite} & \Rightarrow \vec{a} \text{ is a local max} \\ \text{indefinite} & \Rightarrow \vec{a} \text{ is a saddle point} \end{cases}$$

Remark: A critical point which is neither local max nor local min is called a saddle point.

In particular for 2-variable, $\vec{v}^T H \vec{v}$ is of the form

$$g(x,y) = [x\ y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2$$

eg (1) $[x,y]\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 4y^2 > 0, \ \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ is positive definite.

(2) $[x,y]\begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -x^2 - 4y^2 < 0, \ \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$

$\therefore \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$ is negative definite.

(3) $[x,y]\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -x^2 + 4y^2$

If $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $[x,y]\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -0^2 + 4(1)^2 = 4 > 0$

If $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $[x,y]\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -(1)^2 + 4(0)^2 = -1 < 0$

$\therefore \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ is indefinite.

(4) $[x,y]\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 \geq 0, \ \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$

But $[0\ 1]\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \Rightarrow$ not positive definite

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not positive definite, negative definite, nor indefinite.

(5) $$[x,y]\begin{bmatrix}1 & 2\\ 2 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = x^2+4xy+5y^2$$

$$= x^2+4xy+4y^2+y^2$$

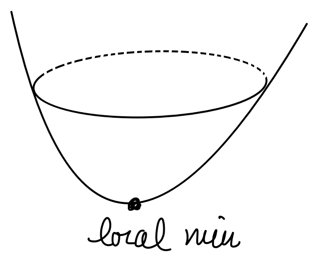
$$= (x+2y)^2+y^2$$

$$> 0 \qquad \forall \begin{bmatrix}x\\ y\end{bmatrix} \in \mathbb{R}^2\setminus\{\vec{0}\}$$

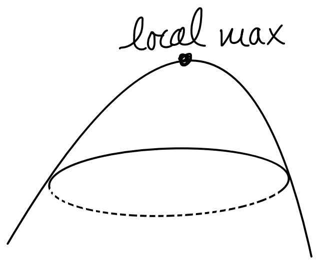
$$\Rightarrow \begin{bmatrix}1 & 2\\ 2 & 5\end{bmatrix} \text{ is positive definite.}$$

<u>Geometrically</u> (locally near the <u>critical point</u>)

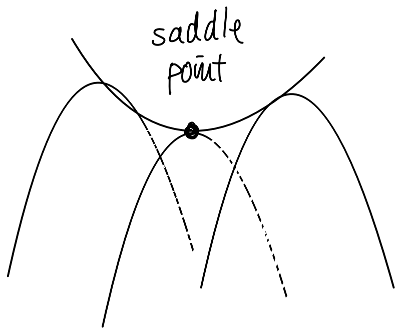
(1) $Hf(\vec{a})$ positive definite



(2) $Hf(\vec{a})$ negative definite



(3) $Hf(\vec{a})$ is indefinite



Thm Let $H = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

Then

$$H \text{ is } \begin{cases} \underline{\text{positive definite}} \iff \det H = ac - b^2 > 0, \ \& \ a > 0 \\ \underline{\text{negative definite}} \iff \det H = ac - b^2 > 0, \ \& \ a < 0 \\ \underline{\text{indefinite}} \iff \det H = ac - b^2 < 0 \end{cases}$$

Pf: Using completing square

If $a \neq 0$, $[x \ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2$

$$= a(x^2 + 2\tfrac{b}{a}xy) + cy^2 = a(x + \tfrac{b}{a}y)^2 + \frac{(ac-b^2)}{a} y^2$$

$$\left(= a\left[(x + \tfrac{b}{a}y)^2 + \frac{ac-b^2}{a^2} y^2 \right] \right)$$

(1) $\det H = ac - b^2 > 0 \iff$

$[x \ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ has the same sign as a.

$\therefore$ The 1st 2 statements are proved.

(2) $\det H = ac - b^2 < 0 \iff$

2 terms have different sign

$\iff \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is indefinite.

If $a = 0 \Rightarrow \det H = ac - b^2 = -b^2 \leq 0$

$[x \ y] \begin{bmatrix} 0 & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2bxy + cy^2$

If $b=0$, then $[x\ y]\begin{bmatrix}0 & b\\ b & c\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = cy^2$ not positive, negative definite nor indefinite

If $b \neq 0$, then
$$\begin{aligned} & 2bxy + cy^2 \\ &= 2by\left(x + \tfrac{c}{2b}y\right) \\ &= \frac{b}{2}\left[\left(y + \left(x + \tfrac{c}{2b}y\right)\right)^2 - \left(y - \left(x + \tfrac{c}{2b}y\right)\right)^2\right] \end{aligned}$$

$\left(\text{using } 4uv = (u+v)^2 - (u-v)^2\right)$

$\Rightarrow$ also indefinite

Still in the case of $\det H = -b^2 < 0 \Leftrightarrow$ indefinite ※

<u>eg1</u> $q(x,y) = 2xy = [x\ y]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$

$\det H = -1 < 0 \Rightarrow$ indefinite

$(a=0)$ $\left(q(x,y) = \frac{1}{2}(x+y)^2 - \frac{1}{2}(x-y)^2\right)$

<u>eg2</u>: $q(x,y) = 17x^2 - 12xy + 8y^2 = [x\ y]\begin{bmatrix}17 & -6\\ -6 & 8\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$

$$\left.\begin{aligned} &\det\begin{bmatrix}17 & -6\\ -6 & 8\end{bmatrix} = 17\cdot 8 - (-6)^2 = 100 > 0, \\ &a = 17 > 0 \end{aligned}\right\} \Rightarrow \text{positive definite}$$

$\left(q(x,y) = 17\left(x - \frac{6}{17}y\right)^2 + \frac{100}{17}y^2\right)$

Then for 2-variable, the 2nd derivative test is

Thm (Second Derivative Test for 2-variables)

Let
- $f : \Omega \to \mathbb{R}$ be C^2, $\Omega \subseteq \mathbb{R}^2$, open
- $(a,b) \in \Omega$ such that $\vec{\nabla} f(a,b) = \vec{0}$

Then

(1) $f_{xx} f_{yy} - f_{xy}^2 > 0$ & $f_{xx} > 0$ at $(a,b) \Rightarrow (a,b)$ is a local min

(2) $f_{xx} f_{yy} - f_{xy}^2 > 0$ & $f_{xx} < 0$ at $(a,b) \Rightarrow (a,b)$ is a local max

(3) $f_{xx} f_{yy} - f_{xy}^2 < 0$ at $(a,b) \Rightarrow (a,b)$ is a saddle point

(4) $f_{xx} f_{yy} - f_{xy}^2 = 0$ at $(a,b) \Rightarrow$ inconclusive.

Remark $\quad f_{xx} f_{yy} - f_{xy}^2 = \det Hf$ (for 2-variables)

eg (1) $\quad f(x,y) = x^3 + y^2$

$$f_{xx} = 6x \quad f_{xy} = 0, \quad f_{yy} = 2$$

$$Hf = \begin{bmatrix} 6x & 0 \\ 0 & 2 \end{bmatrix} \qquad \det Hf = 12x$$

$$\vec{\nabla} f = \vec{0} \Leftrightarrow (3x^2, 2y) = (0,0) \Leftrightarrow (x,y) = (0,0)$$

$$\det Hf(0,0) = 0.$$

$f(t,0) > 0$, $f(-t,0) < 0$ for all $t > 0$ small

$\Rightarrow (0,0)$ is a saddle point

(2) $g(x,y) = x^4 + y^4$ & $h(x,y) = -x^4 - y^4$

Clearly $(0,0)$ is a critical point of both g & h

$$Hg = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix}, \quad Hh = \begin{bmatrix} -12x^2 & 0 \\ 0 & -12y^2 \end{bmatrix}$$

$\Rightarrow$ $\det Hg(0,0) = 0 = \det Hh(0,0)$

$(0,0)$ is a minimum point of g, but $(0,0)$ is a maximum point of h

(1) & (2) $\Rightarrow$ critical point (a,b), can be local max, local min, or saddle in the case when $\det Hf(a,b) = 0$.

eg1 $f(x,y) = 3x^2 - 10xy + 3y^2 + 2x + 2y + 3$

Find and classify critical points of f.

Soln: (f polyomial, always C^2)

$$\vec{\nabla} f = [6x - 10y + 2 \quad -10x + 6y + 2]$$

$$Hf = \begin{bmatrix} 6 & -10 \\ -10 & 6 \end{bmatrix} \quad \text{(a constant matrix)}$$

Critical point:

$$\vec{0} = \vec{\nabla} f \iff \begin{cases} 6x - 10y + 2 = 0 \\ -10x + 6y + 2 = 0 \end{cases}$$

$$\overset{\text{check}}{\iff} (x,y) = (\tfrac{1}{2}, \tfrac{1}{2})$$

$$\det Hf = f_{xx} f_{yy} - f_{xy}^2 = 6 \cdot 6 - (-10)^2 = -64 < 0$$

$\Rightarrow (\frac{1}{2}, \frac{1}{2})$ is a saddle point (by 2^{nd} derivative test)

(No need to check $f_{xx} = 6 > 0$)

※

eg2: $f(x,y) = 3x - x^3 - 3xy^2$

Find and classify critical points of f

Soln (f polynomial, always C^2)

$$\vec{\nabla} f = [3-3x^2-3y^2, \quad -6xy]$$

$$Hf = \begin{bmatrix} -6x & -6y \\ -6y & -6x \end{bmatrix}$$

Critical points :

$$\vec{0} = \vec{\nabla} f \iff \begin{cases} 3-3x^2-3y^2=0 \\ -6xy = 0 \end{cases}$$

$$\overset{\text{check}}{\iff} (x,y) = (0,1), (0,-1), (1,0), (-1,0)$$

(4 critical points)

Critical point	Hf	$f_{xx}f_{yy}-f_{xy}^2$ $= \det Hf$	f_{xx}	classification
$(0,1)$	$\begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$	$-36<0$	No need	saddle
$(0,-1)$	$\begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$	$-36<0$	No need	saddle
$(1,0)$	$\begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$	$36>0$	$-6<0$	local max
$(-1,0)$	$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$	$36>0$	$6>0$	local min

#

eg3 (Inconclusive from 2^{nd} derivative test)

$f(x,y) = x^2+y^4$, $g(x,y)=x^2-y^4$, $h(x,y) = -x^2-y^4$

(at $(0,0)$: local min , saddle , local max)

Soln

$\vec{\nabla} f = [2x \ \ 4y^3]$ $\quad \vec{\nabla} g = [2x \ \ -4y^3]$ $\quad \vec{\nabla} h = [-2x \ \ -4y^3]$

$(\vec{\nabla} f(0,0) = [0, 0] = \vec{\nabla} g(0,0) = \vec{\nabla} h(0,0))$

$Hf = \begin{bmatrix} 2 & 0 \\ 0 & 12y^2 \end{bmatrix}$ $\quad Hg = \begin{bmatrix} 2 & 0 \\ 0 & -12y^2 \end{bmatrix}$ $\quad Hh = \begin{bmatrix} -2 & 0 \\ 0 & -12y^2 \end{bmatrix}$

$Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad Hg = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad Hh = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$

$\det Hf(0,0) = \det Hg(0,0) = \det Hh(0,0) = 0$.

$\therefore$ 2^{nd} derivative test is inclusive.

Higher dimension example

eg: $q(x,y,z) = xy + yz + zx$

has definite sign for $(x,y,z) \neq (0,0,0)$?

Answer: No