eg: $\vec{f}: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \vec{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2-y^2 \\ 2xy \end{bmatrix}$

Clearly $\vec{f}$ is not globally invertible: $\vec{f}\left(\begin{bmatrix} -x \\ -y \end{bmatrix}\right) = \vec{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

it is 2-to-1 (except $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$)

Local inverse?

To check this: $D\vec{f} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$

$$\det D\vec{f} = 4(x^2+y^2) \geq 0 \quad \& \quad "=0" \text{ only if } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $(x,y) \neq (0,0)$, IFT (Inverse Function Thm)

$\Rightarrow \vec{f}$ has a local inverse at (x,y) $(\neq (0,0))$

For instance, let $(x,y) = (1,-1)$

& $\vec{g}(u,v)$ be a local inverse of

$f(x,y)$ "near" $(x,y) = (1,-1)$

$$\left(\text{where } \begin{cases} u = x^2-y^2 \\ v = 2xy \end{cases} \right)$$

$$\vec{f}(1,-1) = (0,-2) \Rightarrow \vec{g}(0,-2) = (1,-1)$$

$$D\vec{g}(0,-2) = \left(D\vec{f}(1,-1)\right)^{-1} = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}^{-1}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \text{(check!)}$$

Explicit calculation of $\vec{g}(u,v)$:

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

near $(x,y) = (1,-1) \Rightarrow x \neq 0 \Rightarrow y = \frac{v}{2x}$

$$\Rightarrow \quad u = x^2 - \left(\frac{v}{2x}\right)^2$$

$$\Rightarrow \quad 4x^4 - 4ux^2 - v^2 = 0$$

$$\Rightarrow \quad x^2 = \frac{4u \pm \sqrt{(-4u)^2 - 4 \cdot 4(-v^2)}}{8}$$

$$= \frac{u \pm \sqrt{u^2+v^2}}{2}$$

Put $(x,y) = (1,-1) \Rightarrow (u,v) = (0,-2)$

$$1^2 = \frac{0 \pm \sqrt{0^2+(-2)^2}}{2}$$

$\Rightarrow$ "$-$" should be rejected

$$\Rightarrow \quad x^2 = \frac{u+\sqrt{u^2+v^2}}{2}$$

$$\Rightarrow \quad x = \sqrt{\frac{u+\sqrt{u^2+v^2}}{2}} \qquad \left(\begin{array}{l} \text{"}-\text{"} \text{ rejected} \\ \text{as } x \text{ near } 1 \end{array}\right)$$

&

$$y = \frac{v}{2x} = \frac{\sqrt{2}\,v}{2\sqrt{u+\sqrt{u^2+v^2}}}$$

$$\therefore \quad \vec{g}(u,v) = \left(\sqrt{\frac{u+\sqrt{u^2+v^2}}{2}} \,,\, \frac{\sqrt{2}\,v}{2\sqrt{u+\sqrt{u^2+v^2}}} \right) \text{ "near" } (0,-2)$$

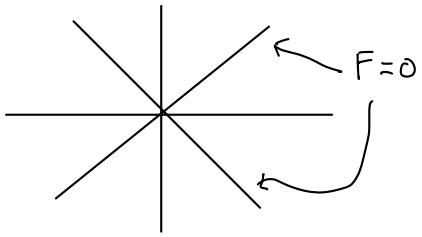
※

Remark: In Implicit Function Thm & Inverse Function Thm, we need to check det. of Jacobian matrix (a sub matrix) is nonzero. In case that the det = 0, we have No conclusion :

eg: **Implicit Function Thm**

$F(x,y) = x^2 - y^2 = 0$

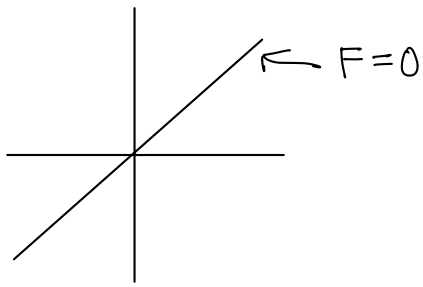
$$\left.\frac{\partial F}{\partial y} = -2y\right|_{(0,0)} = 0$$



y is not locally a function of near (0,0)

$F(x,y) = x^3 - y^3 = 0$

$$\left.\frac{\partial F}{\partial y} = -3y^2\right|_{(0,0)} = 0$$

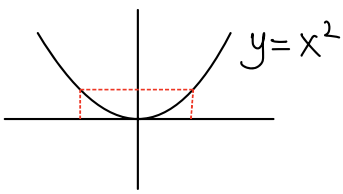


y is locally a function of x near (0,0).

Inverse function Thm

$f(x) = x^2$

$$f'(0) = 0$$

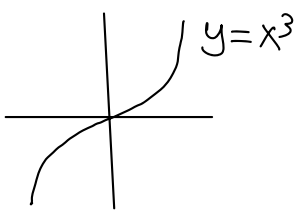


No injective near $x=0$

$\Rightarrow$ no local inverse near $x=0$

$f(x) = x^3$

$$f'(0) = 0$$



$g(y) = y^{\frac{1}{3}}$ is a local inverse near $x=0$.

(End of Term)

Brief review

Basic geometry: vectors, lines, planes,
curves (tangent vectors, arc-length),
open set, closed set,
interior, exterior, boundary

Limit: Definition, Squeeze Thm, Continuity

Partial derivative: 1st and higher order,
Clairaut's Thm (Mixed derivatives thm)
C^k-functions

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mid-term

Differentiability: Linearization, gradient,
directional derivative, total differential

Chain Rule: Jacobian Matrix, normal vector to level set,
Implicit differentiation

Extremum: global max/min on closed & bounded set,
critical points

Taylor's expansion: 2nd derivative test,
Classification of local extremum

Lagrange Multiplier: Constrainted problem, Quadratic constraints

Implicit Function Theorem & Inverse Function Theorem

Final Exam Dec 4 (Wed) 3:30 – 5:30 pm U Gym

Coverage:
- All material in lecture notes, tutorial notes, textbook (Ch 13 & necessary parts of Ch. 10-12) & homework assignments,
- except Implicit Function Thm & Inverse Function Thm (but implicit differentiation is included as application of Chain rules),
- emphasis on those material not included in Midterm.
- 5 questions, answer all. Some are unfamiliar/difficult questions as required by the grade descripter of A range.

(Note: Textbook & assignments contain only basic theory and basic questions.
Past papers may be useful for those material not not included in the Textbook.)

(End)