Implicit Function Theorem

Recall: Implicit differentiation

e.g. $x^2+y^2+z^2=2$ and if $z=z(x,y)$,

then

$$\frac{\partial}{\partial x}(x^2+y^2+z^2)=0 \Rightarrow 2x+2z\frac{\partial z}{\partial x}=0$$

$$\frac{\partial}{\partial y}(x^2+y^2+z^2)=0 \Rightarrow 2y+2z\frac{\partial z}{\partial y}=0$$

If the point (x,y,z) satisfies $z\neq 0$

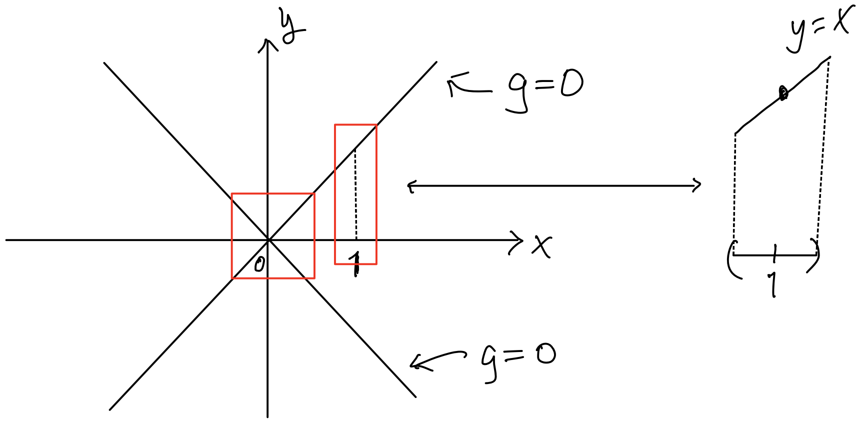
then we have $\frac{\partial z}{\partial x}=-\frac{x}{z}$ & $\frac{\partial z}{\partial y}=-\frac{y}{z}$

Question: If a level set $g(x,y)=c$ (or more generally) is given, can we "solve" the constraint?

i.e. can we find $y=h(x)$ s.t. $g(x,h(x))=c$

or $x=k(y)$ s.t. $g(k(y),y)=c$?

eg1 $g(x,y)=x^2-y^2=0$ $(\Rightarrow x=\pm y)$



Yes, we can solve for $y=h(x)$ near $(x,y)=(1,1)$

eg2 $S: x^2+y^2+z^2=2$ in $\mathbb{R}^3$

Can we solve $z=h(x,y)$ near $(0,1,1)$?

Can we solve $x=k(y,z)$ near $(0,1,1)$?

Observations:

1st question: if $z=h(x,y)$ exists, then

$$\begin{cases} \partial_x(x^2+y^2+z^2)=0 \\ \partial_y(x^2+y^2+z^2)=0 \end{cases} \Rightarrow \begin{cases} \frac{\partial z}{\partial x} = -\frac{x}{z} \\ \frac{\partial z}{\partial y} = -\frac{y}{z} \end{cases}$$ provided that $z\neq 0$.

$$\Rightarrow \quad \frac{\partial z}{\partial x}(0,1,1)=0, \quad \frac{\partial z}{\partial y}(0,1,1)=-1$$

At least, there is no contradiction & we have a hope to solve it!

2nd question: if $x=k(y,z)$ exists

$$\begin{cases} \partial_y(x^2+y^2+z^2)=0 \\ \partial_z(x^2+y^2+z^2)=0 \end{cases} \Rightarrow \begin{cases} 2x\frac{\partial x}{\partial y}+2y=0 \\ 2x\frac{\partial x}{\partial z}+2z=0 \end{cases}$$

At the point $(0,1,1)$, we have $\begin{cases} 0+2=0 \\ 0+2=0 \end{cases}$

which is a contradiction.

So there exists NO $x=k(y,z)$ (which is differentiable) at near the point $(x,y,z)=(0,1,1)$.

General situation (in 3-variables)

$$F(x,y,z) = c$$

If $z = z(x,y)$ (differentiable), then implicit differentiation

$$\begin{aligned} \frac{\partial}{\partial x} : \quad & \begin{cases} \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial x} = 0 \\[2ex] \dfrac{\partial F}{\partial y} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial y} = 0 \end{cases} \\ \frac{\partial}{\partial y} : \quad & \end{aligned}$$

If $F(\vec{a}) = c$ & $\underline{\underline{\frac{\partial F}{\partial z}(\vec{a}) \neq 0}}$, then

$$\begin{bmatrix} \dfrac{\partial z}{\partial x} \\[2ex] \dfrac{\partial z}{\partial y} \end{bmatrix} = -\frac{1}{\frac{\partial F}{\partial z}(\vec{a})} \begin{bmatrix} \dfrac{\partial F}{\partial x}(\vec{a}) \\[2ex] \dfrac{\partial F}{\partial y}(\vec{a}) \end{bmatrix}$$

(at (x_0, y_0) if $\vec{a} = (x_0, y_0, z_0)$)

eg3 (Multiple constraints)

$$\mathcal{C} \begin{cases} x^2 + y^2 + z^2 = 2 \\ x + z = 1 \end{cases}$$

(3-variables, 2 equations
expect $\mathcal{C}$ is "1-dim")

Question: Can we solve $\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y(x) \\ z(x) \end{bmatrix}$?

Observation: If we have $y = y(x)$ & $z = z(x)$, differentiable

then $\frac{d}{dx}\left(x^2 + (y(x))^2 + (z(x))^2 \right) = 0$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} + 2z\frac{dz}{dx} = 0$$

$$\Rightarrow \quad y\frac{dy}{dx} + z\frac{dz}{dx} = -x \qquad \text{——— (1)}$$

and $\frac{d}{dx}(x + z(x)) = 0$

$$\Rightarrow \quad 1 + \frac{dz}{dx} = 0 \qquad \text{——— (2)}$$

$$\therefore \quad \begin{bmatrix} y & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} -x \\ -1 \end{bmatrix}$$

If $\det \begin{bmatrix} y & z \\ 0 & 1 \end{bmatrix} \neq 0$, then one can solve (uniquely) for $\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix}$.

So we have a hope to the existence of $\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y(x) \\ z(x) \end{bmatrix}$.

For instance $(x, y, z) = (0, 1, 1)$ (on $\mathcal{C}$)

$$\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{is solvable}$$

and $$\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \overset{\text{(check)}}{=} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \#$$

<u>In general</u>, given $\mathcal{C}: \begin{cases} F_1(x,y,z) = C_1 \\ F_2(x,y,z) = C_2 \end{cases}$

$$\left(\vec{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad \vec{F}(\vec{x}) = \vec{C}, \quad \vec{F}: \mathbb{R}^3 \to \mathbb{R}^2 \right)$$

Suppose $F_i(a,b,c) = C_i$, $i=1,2$

Assume $y = y(x)$, $z = z(x)$ near (a,b,c) (diff.)

(Implicit differentiation) $\begin{cases} \frac{d}{dx} F_1(x, y(x), z(x)) = 0 \\ \frac{d}{dx} F_2(x, y(x), z(x)) = 0 \end{cases}$

$$\Rightarrow \begin{cases} \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_1}{\partial y}\dfrac{dy}{dx} + \dfrac{\partial F_1}{\partial z}\dfrac{dz}{dx} = 0 \\[2ex] \dfrac{\partial F_2}{\partial x} + \dfrac{\partial F_2}{\partial y}\dfrac{dy}{dx} + \dfrac{\partial F_2}{\partial z}\dfrac{dz}{dx} = 0 \end{cases}$$

i.e. $$\begin{bmatrix} \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\[2ex] \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \end{bmatrix} \begin{bmatrix} \dfrac{dy}{dx} \\[2ex] \dfrac{dz}{dx} \end{bmatrix} = - \begin{bmatrix} \dfrac{\partial F_1}{\partial x} \\[2ex] \dfrac{\partial F_2}{\partial x} \end{bmatrix}$$

$\therefore$ If $\begin{bmatrix} \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\[2ex] \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \end{bmatrix}$ at (a,b,c) is <u>invertiable</u> (i.e. $\det(\ ) \neq 0$)

then $$\begin{bmatrix} \dfrac{dy}{dx} \\[2ex] \dfrac{dz}{dx} \end{bmatrix} = - \begin{bmatrix} \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\[2ex] \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial F_1}{\partial x} \\[2ex] \dfrac{\partial F_2}{\partial x} \end{bmatrix} \quad (\text{at } (a,b,c))$$