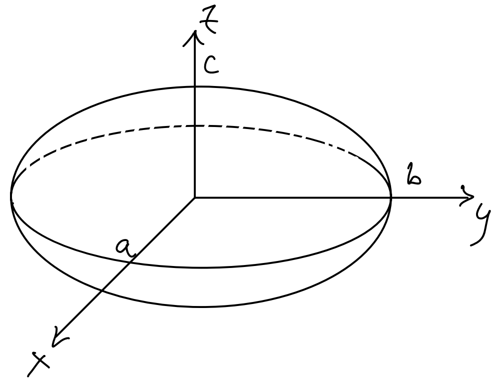
Quadratic Constraint for 3-variables

$$g(x,y,z) = Ax^2+By^2+Cz^2+2Pxy+2Qyz+2Rzx$$
$$+Dx+Ey+Fz+G$$

Same typical examples of $g = \text{const}$.

eg 1 $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Ellipsoid



eg2 $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$ Hyperboloid of 1 sheet

Up to scaling of each variables, graph looks like the graph of

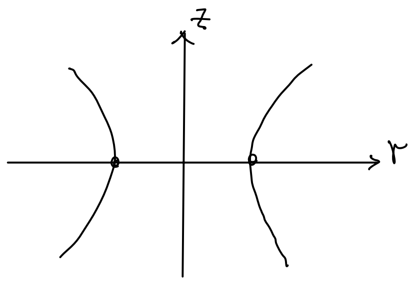
$$x^2+y^2-z^2=1$$

Using polar coordinates on xy-plane,

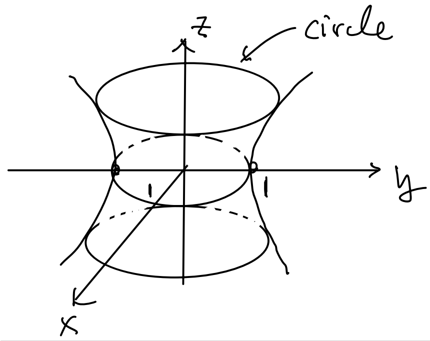
$$\Rightarrow x^2+y^2=r^2$$

$\therefore$ the constraint can be written as

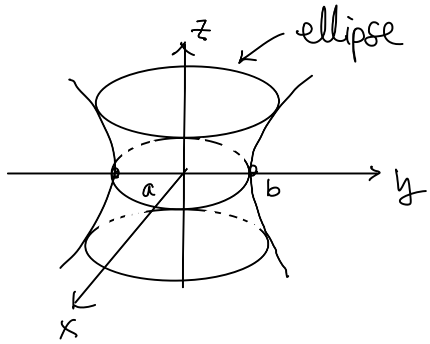
$$r^2-z^2=1$$



same for
all directions
of θ



$\boxed{\text{\underline{Hyperboloid} of 1 sheet}}$



scaling back to

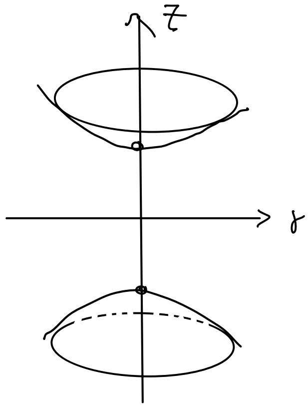
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

<u>eg3</u> $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$ Hyperboloid of 2 sheets

Similarly, after scaling, looks like

$x^2+y^2-z^2=-1 \iff$ in polar $r^2-z^2=-1$

$\iff z^2-r^2=1$



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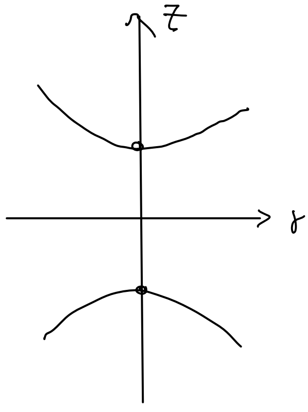


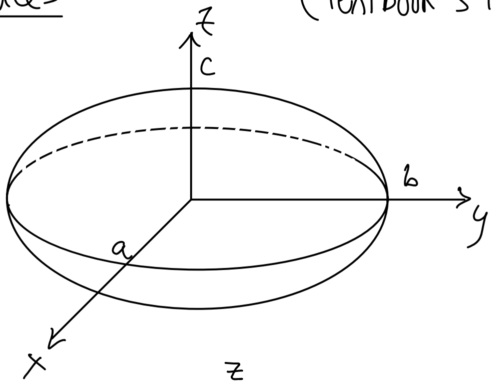
$\boxed{\text{Hyperboloid of 2 sheets}}$

In summary, we have

Graphs of Standard Quadratic Surfaces (Textbook §12.6)

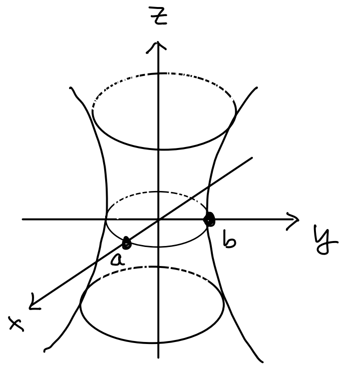
Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



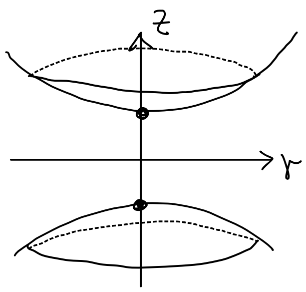
Hyperboloid of 1 sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



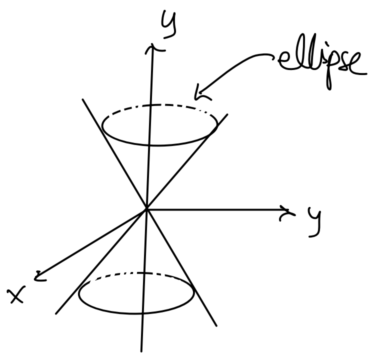
Hyperboloid of 2 sheets

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$



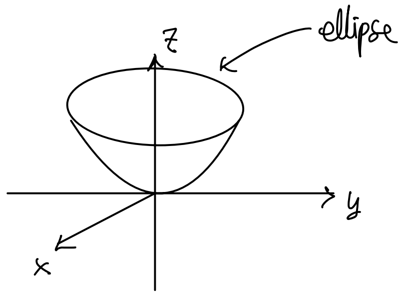
Elliptic Cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$



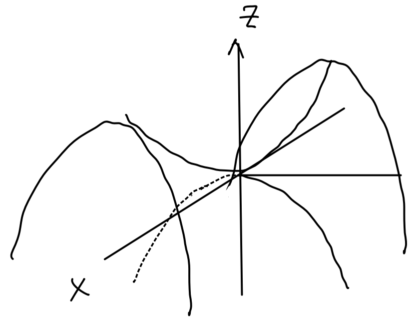
Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



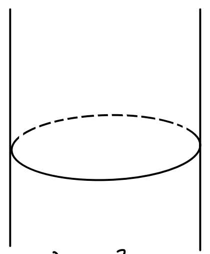
Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

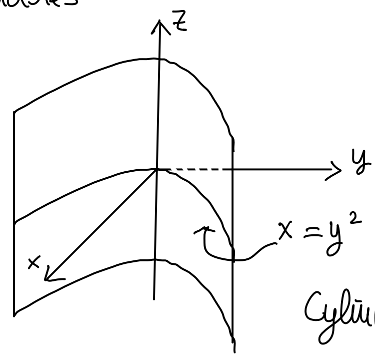


Degenerate to cylinders over conic sections

eg: Equation involve NO z variables



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cylinder of ellipse



and etc.

$x = y^2$

Cylinder of parabola

Other degenerate cases

eg $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0 \quad$ & $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$

$$\boxed{\begin{array}{l}\text{Fact: Any quadratic constraint } g(x,y,z) = c \text{ can be} \\ \text{transformed to one of the standard forms by a} \\ \text{change of coordinates.}\end{array}}$$

Remark: As in 2-variables, only ellipsoid is closed and bounded

Further examples

eg1 Find the point on the ellipse

$$x^2+xy+y^2=9 \qquad \text{(check: it is really a ellipse!)}$$

with maximum x-coordinate.

Soln: Let $f(x,y)=x$ & $g(x,y)=x^2+xy+y^2$

Maximize f under the constraint $g=9$

Consider $F(x,y,\lambda)=x-\lambda(x^2+xy+y^2-9)$

$$\begin{cases} 0=\frac{\partial F}{\partial x}=1-\lambda(2x+y) \\ 0=\frac{\partial F}{\partial y}=-\lambda(x+2y) \\ 0=\frac{\partial F}{\partial \lambda}=-(x^2+xy+y^2-9) \end{cases} \Leftrightarrow \begin{cases} \lambda(2x+y)=1 & \text{---(1)} \\ \lambda(x+2y)=0 & \text{---(2)} \\ x^2+xy+y^2=9 & \text{---(3)} \end{cases}$$

(To be cont'd)

(Cont'd)

$(1) \Rightarrow \lambda \neq 0$

then $(2) \Rightarrow \quad x+2y=0 \Rightarrow \quad x=-2y$

Sub into (3) $\Rightarrow \quad (-2y)^2 + (-2y)y + y^2 = 9$

$\Rightarrow \quad y = \pm\sqrt{3}$ (Check!)

Hence $(x,y) = (-2\sqrt{3}, \sqrt{3})$, $(2\sqrt{3}, -\sqrt{3})$ are the critical points.

Comparing the values $2\sqrt{3} > -2\sqrt{3}$

$\Rightarrow$ max. value of x-coordinates is $2\sqrt{3}$

(at the point $(2\sqrt{3}, -\sqrt{3})$) ※

eg2 Find the point(s) on the hyperboloid

$xy - yz - zx = 3$ (Check: it is really a hyperboloid of 2 sheets)

closest to the origin

Soln Let $f(x,y,z) = x^2+y^2+z^2$

$g(x,y,z) = xy - yz - zx$

Minimize f under constraint $g=3$

Consider $F(x,y,z,\lambda) = x^2+y^2+z^2 - \lambda(xy - yz - zx - 3)$

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = 2x - \lambda(y-z) & \text{———— (1)} \\ 0 = \frac{\partial F}{\partial y} = 2y - \lambda(x-z) & \text{———— (2)} \\ 0 = \frac{\partial F}{\partial z} = 2z + \lambda(x+y) & \text{———— (3)} \\ 0 = \frac{\partial F}{\partial \lambda} = -(xy-yz-zx-3) & \text{———— (4)} \end{cases}$$

If $\lambda = 0$, then (1), (2) & (3) $\Rightarrow$ $x=y=z=0$

contradicting eqt. (4)

So $\lambda \neq 0$.

Then (1), (2) & (3) $\Rightarrow$
$$\begin{cases} y - z = \frac{2}{\lambda}x & \text{—— (5)} \\ x - z = \frac{2}{\lambda}y & \text{—— (6)} \\ x + y = -\frac{2}{\lambda}z & \text{—— (7)} \end{cases}$$

(5)$-$(6) $\Rightarrow$ $y - x = \frac{2}{\lambda}(x-y)$ $\Rightarrow$ $\left(1+\frac{2}{\lambda}\right)(x-y) = 0$ —— (8)

(7)$-$(6) $\Rightarrow$ $y + z = -\frac{2}{\lambda}(z+y)$ $\Rightarrow$ $\left(1+\frac{2}{\lambda}\right)(y+z) = 0$ —— (9)

If $1 + \frac{2}{\lambda} = 0$, i.e. $\lambda = -2$,

then (5), (6), (7) $\Rightarrow$ $x + y - z = 0$

$\Rightarrow$

$$\begin{aligned} 0 = (x+y-z)^2 &= x^2+y^2+z^2+2(xy-yz-xz) \\ &= x^2+y^2+z^2+6 \quad (\text{by } (4)) \end{aligned}$$

which is a contradiction.

$\therefore \quad 1+\frac{2}{\lambda} \neq 0$

Then (8) & (9) $\Rightarrow \quad x=y=-z$

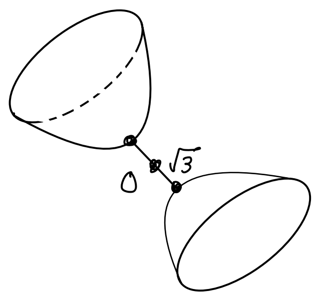
Sub. into (4) $\Rightarrow \quad 3x^2=3 \quad \Rightarrow \quad x=\pm 1$

$\therefore \quad (x,y,z)=\pm(1,1,-1) \quad$ (& $\lambda=1$, if you're interested)

$f(1,1,-1)=f(-1,-1,1)=3$

$\Rightarrow$ closest points are $\pm(1,1,-1)$

with corresponding distance $=\sqrt{3}$



Lagrange Multipliers with multiple Constraints

Let
- $f, g_1, \cdots, g_k : \Omega \to \mathbb{R}$ be C^1 functions, ($\Omega \subseteq \mathbb{R}^n$, open)
- $S = \{\vec{x} \in \Omega : g_i(\vec{x}) = c_i \text{ for } i = 1, \cdots, k\}$

Suppose
- $\vec{a}$ is a local extremum of f on S
- $\vec{\nabla} g_1(\vec{a}), \cdots, \vec{\nabla} g_k(\vec{a})$ are linearly independent vectors

Then

$$\begin{cases} \vec{\nabla} f(\vec{a}) = \sum_{i=1}^{k} \lambda_i \vec{\nabla} g_i(\vec{a}) \\ g_i(\vec{a}) = c_i, \quad i = 1, \cdots, k \end{cases}$$

for some <u>Lagrange multipliers</u> $\lambda_1, \cdots, \lambda_k \in \mathbb{R}$.

Same as 1 constraint,

Finding extrema of $f(\vec{x})$ with constraints $g_i(\vec{x}) = c_i$, $i = 1, \cdots, k$

$$\Updownarrow$$

Finding extrema of $F(\vec{x}, \lambda_1, \cdots, \lambda_k) = f(\vec{x}) - \sum_{i=1}^{k} \lambda_i (g_i(\vec{x}) - c_i)$ without constraint

(but more variables: adding λ_i as new variables) eg1 Maximize $f(x,y,z) = x^2 + 2y - z^2$

on the line $L : \begin{cases} 2x - y = 0 \\ y + z = 0 \end{cases}$ in $\mathbb{R}^3$

(Given that maximum exists)

Soln Let $g_1(x,y,z) = 2x - y$

$g_2(x,y,z) = y + z$

Maximize f subject to constraints $\begin{cases} g_1 = 0 \\ g_2 = 0 \end{cases}$

[f is 2-degree poly, g_1, g_2 are degree 1 polynomials $\Rightarrow f, g_1, g_2$ are C^1]

$$\left.\begin{aligned} \vec{\nabla} g_1 &= (2 \quad -1 \quad 0) \\ \vec{\nabla} g_2 &= (0 \quad 1 \quad 1) \end{aligned}\right\} \text{are linearly independent (prove it!)}$$

Consider

$$F(x,y,z,\lambda_1,\lambda_2) = x^2 + 2y - z^2 - \lambda_1(2x - y) - \lambda_2(y + z)$$

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = 2x - 2\lambda_1 \\ 0 = \frac{\partial F}{\partial y} = 2 + \lambda_1 - \lambda_2 \\ 0 = \frac{\partial F}{\partial z} = -2z - \lambda_2 \\ 0 = \frac{\partial F}{\partial \lambda_1} = -(2x - y) \\ 0 = \frac{\partial F}{\partial \lambda_2} = -(y + z) \end{cases} \Leftrightarrow \begin{cases} x = \lambda_1 & \text{--- (1)} \\ \lambda_2 = \lambda_1 + 2 & \text{--- (2)} \\ \lambda_2 = -2z & \text{--- (3)} \\ 2x = y & \text{--- (4)} \\ y = -z & \text{--- (5)} \end{cases}$$

(1) & (3) sub into (2)

$$-2z = x+2 \qquad \text{———— } (6)$$

(4) & (5) $\Rightarrow$ $\quad 2x = y = -z \qquad$ ———— (7)

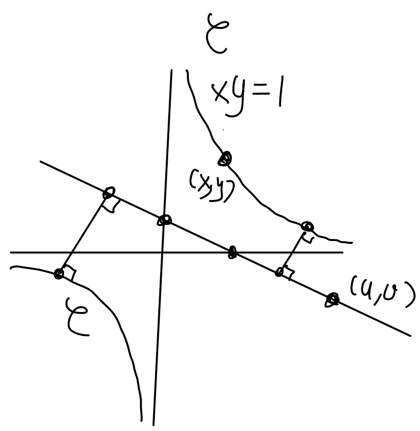
sub into (6) $\qquad 4x = x+2 \Rightarrow x = \frac{2}{3}$

sub into (7) $\Rightarrow \qquad y = \frac{4}{3}, \quad z = -\frac{4}{3}$

$\Rightarrow$ max occurs at $\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$

with value $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \left(\frac{2}{3}\right)^2 + 2\left(\frac{4}{3}\right) - \left(\frac{4}{3}\right)^2$

(check!) $= \frac{4}{3}$ #



eg2 Find the distance between the hyperbola $\mathcal{E}: xy = 1$ and the line $L: x+4y = \frac{15}{8}$

Soln: Let

$$f(x,y,u,v) = (x-u)^2 + (y-v)^2$$

Minimize f under constraints

$$g_1(x,y,u,v) = xy = 1$$

$$g_2(x,y,u,v) = u+4v = \frac{15}{8}$$

$$\vec{\nabla} g_1 = [\, y \quad x \quad 0 \quad 0 \,]$$
$$\vec{\nabla} g_2 = [\, 0 \quad 0 \quad 1 \quad 4 \,]$$

$\vec{\nabla} g_1$ & $\vec{\nabla} g_2$ are linearly independent

$\iff (x, y) \neq (0,0)$ (Can you prove it ?)

Consider

$$F(x,y,u,v,\lambda_1,\lambda_2) = (x-u)^2+(y-v)^2-\lambda_1(xy-1)-\lambda_2\left(u+4v-\frac{15}{8}\right)$$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x} = 2(x-u) - \lambda_1 y & \text{———— (1)} \\ 0 = \dfrac{\partial F}{\partial y} = 2(y-v) - \lambda_1 x & \text{———— (2)} \\ 0 = \dfrac{\partial F}{\partial u} = -2(x-u) - \lambda_2 & \text{———— (3)} \\ 0 = \dfrac{\partial F}{\partial v} = -2(y-v) - 4\lambda_2 & \text{———— (4)} \\ 0 = \dfrac{\partial F}{\partial \lambda_1} = -(xy-1) & \text{———— (5)} \\ 0 = \dfrac{\partial F}{\partial \lambda_2} = -\left(u+4v-\dfrac{15}{8}\right) & \text{———— (6)} \end{cases}$$

Case 1 If $\lambda_1 = 0$ or $\lambda_2 = 0$, then

$x = u$ & $y = v$

sub into (6) $\Rightarrow$ $x = \frac{15}{8} - 4y$

sub into (5) $\Rightarrow$ $\left(\frac{15}{8} - 4y\right)y = 1$

$4y^2 - \frac{15}{8}y + 1 = 0$ has no (real) solution

Case 2 $\lambda_1 \neq 0$ & $\lambda_2 \neq 0$.

Then (3) & (4) $\Rightarrow$

$$\left.\begin{array}{l} \dfrac{x-u}{y-v} = \dfrac{1}{4} \\ \\ \& \quad (1) \& (2) \Rightarrow \\ \\ \dfrac{x-u}{y-v} = \dfrac{y}{x} \end{array}\right\} \Rightarrow x = 4y$$

sub. into (5) $\quad (4y)y = 1 \Rightarrow y = \pm\frac{1}{2}$

$\therefore \ (x,y) = \pm(2,\frac{1}{2}) \qquad (\neq (0,0))$

Then for $(2,\frac{1}{2})$, $\dfrac{2-u}{\frac{1}{2}-v} = \dfrac{1}{4} \Rightarrow$

together (6)

$$\left.\begin{array}{l} 4u - v = \frac{15}{2} \\ u + 4v = \frac{15}{8} \end{array}\right\}$$

$\Rightarrow (u,v) = (\frac{15}{8}, 0)$ (Check!)

Similarly for $(-2,-\frac{1}{2})$, we have $(u,v) = (-\frac{225}{136}, \frac{15}{17})$ (Ex!)

Comparing the values $\quad f(2,\frac{1}{2},\frac{15}{8},0) = \frac{17}{64} \left(= (\text{dist})^2\right)$ (check!)

$f(-2,-\frac{1}{2},-\frac{225}{136},\frac{15}{17}) = \cdots > \frac{17}{64}$

check

$\Rightarrow$ distance between $\mathcal{E}$ and $L = \frac{\sqrt{17}}{8}$ (Check) ※