MATH2010 Advanced Calculus I

Solution to Homework 9

§13.7

Q2

Solution.
$f_x(x,y) = 2y - 10x + 4 = 0$
$f_y(x,y) = 2x - 4y + 4 = 0$
Hence, $(x,y) = (\frac{2}{3}, \frac{4}{3})$ is a critical point.
$f_{xx}(x,y) = -10, f_{yy}(x,y) = -4, f_{xy}(x,y) = 2$
$f_{xx}(\frac{2}{3}, \frac{4}{3}) = -10, f_{yy}(\frac{2}{3}, \frac{4}{3}) = -4, f_{xy}(\frac{2}{3}, \frac{4}{3}) = 2$
$f_{xx} = -10 < 0$ and $f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$
Therefore, f has a local maximum at $(\frac{2}{3}, \frac{4}{3})$. $f(\frac{2}{3}, \frac{4}{3}) = 0$. □

Q6

Solution.
$f_x(x,y) = 2x - 4y = 0$
$f_y(x,y) = -4x + 2y + 6 = 0$
Hence, $(x,y) = (2,1)$ is a critical point.
$f_{xx}(x,y) = 2, f_{yy}(x,y) = 2, f_{xy}(x,y) = -4$
$f_{xx}(2,1) = 2, f_{yy}(2,1) = 2, f_{xy}(2,1) = -4$
$f_{xx}f_{yy} - f_{xy}^2 = -12 < 0$
Therefore, f has a saddle point at $(2,1)$. $f(2,1) = 5$. □

Q12

Solution.
$f_x(x,y) = -\frac{2x}{3(x^2+y^2)^{2/3}} = 0$
$f_y(x,y) = -\frac{2y}{3(x^2+y^2)^{2/3}} = 0$
There are no solutions.
For $f_x(x,y), f_y(x,y)$ do not exist, $(x,y) = (0,0)$.
Hence, $(0,0)$ is a critical point.
$f(0,0) = 1$
Since $x^2 + y^2 \geq 0$, $f(x,y) \leq 1$ for all (x,y).
Therefore, f has a local maximum at $(0,0)$. $f(0,0) = 1$. □

Q16

Solution.
$f_x(x,y) = 3x^2 + 6x = 0 \Rightarrow x = 0, -2$
$f_y(x,y) = 3y^2 - 6y = 0 \Rightarrow y = 0, 2$
Hence, $(0,0), (0,2), (-2,0), (-2,2)$ are critical points.
$f_{xx}(x,y) = 6x + 6, f_{yy}(x,y) = 6y - 6, f_{xy}(x,y) = 0$
For $(x,y) = (0,0)$, $f_{xx}f_{yy} - f_{xy}^2 = (6)(-6) - 0 = -36 < 0$. f has a saddle point at $(0,0)$. $f(0,0) = -8$.
For $(x,y) = (0,2)$, $f_{xx}f_{yy} - f_{xy}^2 = (6)(6) - 0 = 36 > 0, f_{xx} = 6 > 0$. f has a local minimum at $(0,2)$. $f(0,2) = -12$.

For $(x,y) = (-2,0)$, $f_{xx}f_{yy} - f_{xy}^2 = (-6)(-6) - 0 = 36 > 0, f_{xx} = -6 < 0$. f has a local maximum at $(-2,0)$. $f(-2,0) = -4$.
For $(x,y) = (-2,2)$, $f_{xx}f_{yy} - f_{xy}^2 = (-6)(6) - 0 = -36 < 0$. f has a saddle point at $(-2,2)$. $f(-2,2) = -8$.

□

Q22

Solution.
$f_x(x,y) = -\frac{1}{x^2} + y = 0$
$f_y(x,y) = x - \frac{1}{y^2} = 0$
Hence, $(x,y) = (1,1)$ is a critical point.
$f_{xx}(x,y) = \frac{2}{x^3}, f_{yy}(x,y) = \frac{2}{y^3}, f_{xy}(x,y) = 1$
$f_{xx}(1,1) = 2, f_{yy}(1,1) = 2, f_{xy}(1,1) = 1$
$f_{xx} = 2 > 0$ and $f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$
Therefore, f has a local minimum at $(1,1)$. $f(1,1) = 3$. □

Q28

Solution.
$f_x(x,y) = e^x(x^2 - y^2) + e^x(2x) = e^x(x^2 + 2x - y^2) = 0$
$f_y(x,y) = -2ye^x = 0$
Hence, $(x,y) = (0,0), (-2,0)$ are critical points.
$f_{xx}(x,y) = e^x(x^2 + 2x - y^2) + e^x(2x+2) = e^x(x^2 + 4x + 2 - y^2), f_{yy}(x,y) = -2e^x, f_{xy}(x,y) = -2ye^x$
For $(x,y) = (0,0)$, $f_{xx}f_{yy} - f_{xy}^2 = (2)(-2) - 0 = -4 < 0$. f has a saddle point at $(0,0)$. $f(0,0) = 0$.
For $(x,y) = (-2,0)$, $f_{xx}f_{yy} - f_{xy}^2 = (-\frac{2}{e^2})(-\frac{2}{e^2}) - 0 = \frac{4}{e^4} > 0, f_{xx} = -\frac{2}{e^2} < 0$. f has a local maximum at $(-2,0)$. $f(-2,0) = \frac{4}{e^2}$.

□

Q30

Solution.
$f_x(x,y) = \frac{1}{x+y} + 2x = 0$
$f_y(x,y) = \frac{1}{x+y} - 1 = 0$
Hence, $(x,y) = (-\frac{1}{2}, \frac{3}{2})$ is a critical point.
$f_{xx}(x,y) = -\frac{1}{(x+y)^2} + 2, f_{yy}(x,y) = -\frac{1}{(x+y)^2}, f_{xy}(x,y) = -\frac{1}{(x+y)^2}$
$f_{xx}(-\frac{1}{2}, \frac{3}{2}) = -1 + 2 = 1, f_{yy}(-\frac{1}{2}, \frac{3}{2}) = -1, f_{xy}(-\frac{1}{2}, \frac{3}{2}) = -1$
$f_{xx}f_{yy} - f_{xy}^2 = (1)(-1) - (-1)^2 = -2 < 0$
Therefore, f has a saddle point at $(-\frac{1}{2}, \frac{3}{2})$. $f(-\frac{1}{2}, \frac{3}{2}) = \ln(1) + \frac{1}{4} - \frac{3}{2} = -\frac{5}{4}$. □

Q44

Solution.

(a.) $f(0,0) = 0$
$f(x,y) = x^2y^2 \geq 0$ for all (x,y)
Minimum at $(0,0)$

(b.) $f(0,0) = 1 - 0 = 1$
Since $x^2y^2 \geq 0$, $f(x,y) = 1 - x^2y^2 \leq 1$ for all (x,y)
Maximum at $(0,0)$

(c.) $f(0,0) = 0$
Since for $y \neq 0$, $y^2 > 0$, then $f(x,y) = xy^2 < 0$ for $x < 0$ and > 0 for $x > 0$.
Neither at $(0,0)$

(d.) $f(0,0) = 0$
Since for $y \neq 0$, $y^2 > 0$, then $f(x,y) = x^3y^2 < 0$ for $x < 0$ and > 0 for $x > 0$.
Neither at $(0,0)$

(e.) $f(0,0) = 0$
$f(x,y) = x^3y^3 < 0$ for $x < 0, y > 0$ and > 0 for $x > 0, y > 0$.
Neither at $(0,0)$

(f.) $f(0,0) = 0$
$f(x,y) = x^4y^4 \geq 0$ for all (x,y)
Minimum at $(0,0)$

□

Q46

Solution.
$f_x(x,y) = 2x + ky, f_y(x,y) = kx + 2y$
$f_x(0,0) = 0, f_y(0,0) = 0$
Hence, $(0,0)$ is a critical point.
$f_{xx}(x,y) = 2, f_{yy}(x,y) = 2, f_{xy}(x,y) = k$
$f_{xx}(0,0) = 2, f_{yy}(0,0) = 2, f_{xy}(0,0) = k$
$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - k^2 = 4 - k^2$
f will have a saddle point at $(0,0)$ if $4 - k^2 < 0 \Rightarrow k < -2$ or $k > 2$.
f will have a local minimum at $(0,0)$ if $4 - k^2 > 0 \Rightarrow -2 < k < 2$.
The Second Derivative Test is inconclusive if $4 - k^2 = 0 \Rightarrow k = -2, 2$. □

Q48

Solution.
If $f_{xx}(a,b)$ and $f_{yy}(a,b)$ differ in sign, then $f_{xx}(a,b)f_{yy}(a,b) < 0$.
Since $f_{xy}^2 \geq 0$, then $f_{xx}f_{yy} - f_{xy}^2 < 0$.
Therefore, f has a saddle point at the critical point (a,b) by the second derivative test. □