MATH2010 Advanced Calculus I

Solution to Homework 8

§13.5

Q42

Solution.
Let $f(x, y) = x^2 + xy + y^2$.
$f_x(x, y) = 2x + y, f_y(x, y) = x + 2y$
$f_x(2, -1) = 4 - 1 = 3, f_y(2, -1) = 2 - 2 = 0$
The line that is perpendicular to the graph of the given equation at $(2, -1)$ is given by
$(x, y) = (2, -1) + t(3, 0) = (2 + 3t, -1), t \in \mathbb{R}$. □

Q44

Solution.
$x^3 - xy^2 - z = 0$
Let $f(x, y, z) = x^3 - xy^2 - z$.
$f_x(x, y, z) = 3x^2 - y^2, f_y(x, y, z) = -2xy, f_z(x, y, z) = -1$
$f_x(-1, 1, 0) = 3 - 1 = 2, f_y(-1, 1, 0) = 2, f_z(-1, 1, 0) = -1$
The line that is perpendicular to the graph of the given equation at $(-1, 1, 0)$ is given by
$(x, y, z) = (-1, 1, 0) + t(2, 2, -1) = (-1 + 2t, 1 + 2t, -t), t \in \mathbb{R}$. □

§13.7

Q32

Solution.
$D(x, y) = x^2 - xy + y^2 + 1$

For interior points:
$D_x(x, y) = 2x - y = 0$
$D_y(x, y) = -x + 2y = 0$
Hence, $(x, y) = (0, 0)$ which is not an interior point of the region.

For boundary points:
On $x = 0,\ 0 \le y \le 4,\ D(x, y) = D(0, y) = y^2 + 1$
$D'(0, y) = 2y = 0 \quad \Rightarrow \quad y = 0 \quad \Rightarrow \quad (x, y) = (0, 0)$ which is an endpoint
At the endpoints: $D(0, 0) = 1, D(0, 4) = 17$

On $y = 4,\ 0 \le x \le 4,\ D(x, y) = D(x, 4) = x^2 - 4x + 17$
$D'(x, 4) = 2x - 4 = 0 \quad \Rightarrow \quad x = 2 \quad \Rightarrow \quad (x, y) = (2, 4)$
$D(2, 4) = 13$
At the endpoints: $D(0, 4) = 17, D(4, 4) = 17$

On $y = x,\ 0 \le x \le 4, 0 \le y \le 4,\ D(x, y) = D(x, x) = x^2 + 1$
$D'(x, x) = 2x = 0 \quad \Rightarrow \quad x = 0 \quad \Rightarrow \quad (x, y) = (0, 0)$ which is an endpoint
Endpoint values have been found above.

Comparing the values at the above points, we have
Absolute maximum = 17 at $(0,4)$ and $(4,4)$,
Absolute minimum = 1 at $(0,0)$. □

Q34

Solution.
$T(x,y) = x^2 + xy + y^2 - 6x$

For interior points:
$T_x(x,y) = 2x + y - 6 = 0$
$T_y(x,y) = x + 2y = 0$
Hence, $(x,y) = (4,-2)$ is an interior critical point.
$T(4,-2) = -12$

For boundary points:
On $x = 0$, $-3 \le y \le 3$, $T(x,y) = T(0,y) = y^2$
$T'(0,y) = 2y = 0 \quad \Rightarrow \quad y = 0 \quad \Rightarrow \quad (x,y) = (0,0)$
$T(0,0) = 0$
At the endpoints: $T(0,-3) = 9, T(0,3) = 9$

On $x = 5$, $-3 \le y \le 3$, $T(x,y) = T(5,y) = y^2 + 5y - 5$
$T'(5,y) = 2y + 5 = 0 \quad \Rightarrow \quad y = -\frac{5}{2} \quad \Rightarrow \quad (x,y) = (5, -\frac{5}{2})$
$T(5,-\frac{5}{2}) = -\frac{45}{4}$
At the endpoints: $T(5,-3) = -11, T(5,3) = 19$

On $y = -3$, $0 \le x \le 5$, $T(x,y) = T(x,-3) = x^2 - 9x + 9$
$T'(x,-3) = 2x - 9 = 0 \quad \Rightarrow \quad x = \frac{9}{2} \quad \Rightarrow \quad (x,y) = (\frac{9}{2}, -3)$
$T(\frac{9}{2},-3) = -\frac{45}{4}$
Endpoint values have been found above.

On $y = 3$, $0 \le x \le 5$, $T(x,y) = T(x,3) = x^2 - 3x + 9$
$T'(x,3) = 2x - 3 = 0 \quad \Rightarrow \quad x = \frac{3}{2} \quad \Rightarrow \quad (x,y) = (\frac{3}{2}, 3)$
$T(\frac{3}{2},3) = \frac{27}{4}$
Endpoint values have been found above.

Comparing the values at the above points, we have
Absolute maximum = 19 at $(5,3)$,
Absolute minimum = -12 at $(4,-2)$. □

Q37

Solution.
$f(x,y) = (4x - x^2)\cos y$

For interior points:
$f_x(x,y) = (4 - 2x)\cos y = 0$
$f_y(x,y) = -(4x - x^2)\sin y = 0$
Hence, $(x,y) = (2,0)$ is an interior critical point.
$f(2,0) = 4$

For boundary points:
On $x = 1$, $-\pi/4 \le y \le \pi/4$, $f(x,y) = f(1,y) = 3\cos y$
$f'(1,y) = -3\sin y = 0 \quad \Rightarrow \quad y = 0 \quad \Rightarrow \quad (x,y) = (1,0)$

$f(1,0) = 3$
At the endpoints: $f(1,-\pi/4) = \frac{3\sqrt{2}}{2}, f(1,\pi/4) = \frac{3\sqrt{2}}{2}$

On $x = 3$, $-\pi/4 \le y \le \pi/4$, $f(x,y) = f(3,y) = 3\cos y$
$f'(3,y) = -3\sin y = 0 \quad \Rightarrow \quad y = 0 \quad \Rightarrow \quad (x,y) = (3,0)$
$f(3,0) = 3$
At the endpoints: $f(3,-\pi/4) = \frac{3\sqrt{2}}{2}, f(3,\pi/4) = \frac{3\sqrt{2}}{2}$

On $y = -\pi/4$, $1 \le x \le 3$, $f(x,y) = f(x,-\pi/4) = \frac{\sqrt{2}}{2}(4x - x^2)$
$f'(x,-\pi/4) = \frac{\sqrt{2}}{2}(4-2x) = 0 \quad \Rightarrow \quad x = 2 \quad \Rightarrow \quad (x,y) = (2,-\pi/4)$
$f(2,-\pi/4) = 2\sqrt{2}$
Endpoint values have been found above.

On $y = \pi/4$, $1 \le x \le 3$, $f(x,y) = f(x,\pi/4) = \frac{\sqrt{2}}{2}(4x - x^2)$
$f'(x,\pi/4) = \frac{\sqrt{2}}{2}(4-2x) = 0 \quad \Rightarrow \quad x = 2 \quad \Rightarrow \quad (x,y) = (2,\pi/4)$
$f(2,\pi/4) = 2\sqrt{2}$
Endpoint values have been found above.

Comparing the values at the above points, we have
Absolute maximum = 4 at $(2,0)$,
Absolute minimum = $\frac{3\sqrt{2}}{2}$ at $(1,-\pi/4),(1,\pi/4),(3,-\pi/4)$ and $(3,\pi/4)$.

□

Q38

Solution.
$f(x,y) = 4x - 8xy + 2y + 1$

For interior points:
$f_x(x,y) = 4 - 8y = 0$
$f_y(x,y) = -8x + 2 = 0$
Hence, $(x,y) = (\frac{1}{4},\frac{1}{2})$ is an interior critical point.
$f(\frac{1}{4},\frac{1}{2}) = 2$

For boundary points:
On $x = 0$, $0 \le y \le 1$, $f(x,y) = f(0,y) = 2y + 1$
$f'(0,y) = 2 \ne 0 \quad \Rightarrow$ no interior critical points
At the endpoints: $f(0,0) = 1, f(0,1) = 3$

On $y = 0$, $0 \le x \le 1$, $f(x,y) = f(x,0) = 4x + 1$
$f'(x,0) = 4 \ne 0 \quad \Rightarrow$ no interior critical points
At the endpoints: $f(0,0) = 1, f(1,0) = 5$

On $x + y = 1$, $0 \le x \le 1, 0 \le y \le 1$, $f(x,y) = f(x,-x+1) = 4x + 8x^2 - 8x - 2x + 2 + 1 = 8x^2 - 6x + 3$
$f'(x,-x+1) = 16x - 6 = 0 \quad \Rightarrow \quad x = \frac{3}{8} \quad \Rightarrow \quad (x,y) = (\frac{3}{8},\frac{5}{8})$
$f(\frac{3}{8},\frac{5}{8}) = \frac{15}{8}$
Endpoint values have been found above.

Comparing the values at the above points, we have
Absolute maximum = 5 at $(1,0)$,
Absolute minimum = 1 at $(0,0)$.

□

Q42

Solution.
$f(x,y) = xy + 2x - \ln x^2 y$

$f_x(x,y) = y + 2 - \frac{1}{x^2 y} 2xy = y + 2 - \frac{2}{x} = 0$
$f_y(x,y) = x - \frac{1}{x^2 y} x^2 = x - \frac{1}{y} = 0$
Hence, $(x,y) = (\frac{1}{2}, 2)$ is the critical point in the open first quadrant.

$f_{xx}(x,y) = \frac{2}{x^2}, f_{xy}(x,y) = 1, f_{yx}(x,y) = 1, f_{yy}(x,y) = \frac{1}{y^2}$
$f_{xx}(\frac{1}{2},2) = 8, f_{xy}(\frac{1}{2},2) = 1, f_{yx}(\frac{1}{2},2) = 1, f_{yy}(\frac{1}{2},2) = \frac{1}{4}$
Since $f_{xx}(\frac{1}{2},2) = 8 > 0$ and $\det Hf(\frac{1}{2},2) = \det \begin{pmatrix} 8 & 1 \\ 1 & \frac{1}{4} \end{pmatrix} = 1 > 0, (\frac{1}{2},2)$is a local minimum. □

Q45

Solution.
$f(x,y) = x^2 + kxy + y^2$

If $k = 0, f(x,y) = x^2 + y^2$
$f_x(x,y) = 2x = 0$
$f_y(x,y) = 2y = 0$
Hence, $(x,y) = (0,0)$ is a critical point.

If $k \neq 0$,
$f_x(x,y) = 2x + ky$
$f_y(x,y) = kx + 2y$
Since $f_x(0,0) = 0 + 0 = 0$ and $f_y(0,0) = 0 + 0 = 0$, $(0,0)$ is a critical point.

$(0,0)$ is a critical point no matter what value the constant k has. □

Q60

Solution.
$f(x,y) = x^2 + y^2 + 2xy - x - y + 1$

(a) For interior points:
$f_x(x,y) = 2x + 2y - 1 = 0$
$f_y(x,y) = 2y + 2x - 1 = 0$
Hence, $2x + 2y = 1$.
Then, $x + y = \frac{1}{2}, f(x,y) = (x+y)^2 - (x+y) + 1 = \frac{3}{4}$

For boundary points:
On $x = 0$, $0 \leq y \leq 1$, $f(x,y) = f(0,y) = y^2 - y + 1$
$f'(0,y) = 2y - 1 = 0 \quad \Rightarrow \quad y = \frac{1}{2} \quad \Rightarrow \quad (x,y) = (0, \frac{1}{2})$
$f(0, \frac{1}{2}) = \frac{3}{4}$
At the endpoints: $f(0,0) = 1, f(0,1) = 1$

On $x = 1$, $0 \leq y \leq 1$, $f(x,y) = f(1,y) = y^2 + y + 1$
$f'(1,y) = 2y + 1 = 0 \quad \Rightarrow \quad y = -\frac{1}{2}$ but $(x,y) = (1, -\frac{1}{2})$ is outside the square domain
At the endpoints: $f(1,0) = 1, f(1,1) = 3$

On $y = 0$, $0 \leq x \leq 1$, $f(x,y) = f(x,0) = x^2 - x + 1$
$f'(x,0) = 2x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{2} \quad \Rightarrow \quad (x,y) = (\frac{1}{2}, 0)$
$f(\frac{1}{2}, 0) = \frac{3}{4}$

Endpoint values have been found above.

On $y = 1$, $0 \le x \le 1$, $f(x, y) = f(x, 1) = x^2 + x + 1$
$f'(x, 1) = 2x + 1 = 0 \quad \Rightarrow \quad x = -\frac{1}{2}$ but $(x, y) = (-\frac{1}{2}, 1)$ is outside the square domain
Endpoint values have been found above.

By comparing the above values, f has an absolute minimum when $2x + 2y = 1$ in this square.
The absolute minimum value is $\frac{3}{4}$.

(b) Absolute maximum value = 3 at $(1, 1)$

□

Q64

Solution.

(a) $f(x, y) = 2x + 3y$
For $x^2/9 + y^2/4 = 1$, let $x = 3\cos t$ and $y = 2\sin t$. Then,
$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 2(-3\sin t) + 3(2\cos t) = -6\sin t + 6\cos t = 0 \Rightarrow \cos t = \sin t$$

(i) For interior points, $t = \pi/4$
$x = 3/\sqrt{2}, y = \sqrt{2}$
$f(3/\sqrt{2}, \sqrt{2}) = 3\sqrt{2} + 3\sqrt{2} = 6\sqrt{2}$

At the endpoints, $f(-3, 0) = -6$, $f(3, 0) = 6$.

Absolute maximum $= f(3/\sqrt{2}, \sqrt{2}) = 6\sqrt{2}$ when $t = \pi/4$
Absolute minimum $= f(-3, 0) = -6$ when $t = \pi$.

(ii) For interior points, $t = \pi/4$
$x = 3/\sqrt{2}, y = \sqrt{2}$
$f(3/\sqrt{2}, \sqrt{2}) = 3\sqrt{2} + 3\sqrt{2} = 6\sqrt{2}$

At the endpoints, $f(0, 2) = 6$, $f(3, 0) = 6$.

Absolute maximum $= f(3/\sqrt{2}, \sqrt{2}) = 6\sqrt{2}$ when $t = \pi/4$
Absolute minimum $= f(3, 0) = 6, f(0, 2) = 6$ when $t = 0, \pi/2$.

(b) $g(x, y) = xy$
For $x^2/9 + y^2/4 = 1$, let $x = 3\cos t$ and $y = 2\sin t$. Then,
$$\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = (2\sin t)(-3\sin t) + (3\cos t)(2\cos t) = -6\sin^2 t + 6\cos^2 t = 0 \Rightarrow \cos^2 t = \sin^2 t$$

(i) For interior points, $t = \pi/4$ or $t = 3\pi/4$
$g(3/\sqrt{2}, \sqrt{2}) = 3$ when $t = \pi/4$
$g(-3/\sqrt{2}, \sqrt{2}) = -3$ when $t = 3\pi/4$

At the endpoints, $g(-3, 0) = 0$, $g(3, 0) = 0$.

Absolute maximum $= g(3/\sqrt{2}, \sqrt{2}) = 3$ when $t = \pi/4$
Absolute minimum $= g(-3/\sqrt{2}, \sqrt{2}) = -3$ when $t = 3\pi/4$

(ii) For interior points, $t = \pi/4$
$g(3/\sqrt{2}, \sqrt{2}) = 3$ when $t = \pi/4$

At the endpoints, $g(0,2) = 0$, $g(3,0) = 0$.

Absolute maximum $= g(3/\sqrt{2}, \sqrt{2}) = 3$ when $t = \pi/4$
Absolute minimum $= g(3,0) = 0$, $g(0,2) = 0$ when $t = 0, \pi/2$.

(c) $h(x,y) = x^2 + 3y^2$
For $x^2/9 + y^2/4 = 1$, let $x = 3\cos t$ and $y = 2\sin t$. Then,

$$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} = (6\cos t)(-3\sin t) + (12\sin t)(2\cos t) = -18\sin t\cos t + 24\sin t\cos t = 0 \Rightarrow \sin t\cos t = 0$$

(i) $t = 0, \pi/2, \pi$
$h(3,0) = 9$
$h(0,2) = 12$
$h(-3,0) = 9$

Absolute maximum $= h(0,2) = 12$ when $t = \pi/2$
Absolute minimum $= h(3,0) = 9$, $h(-3,0) = 9$ when $t = 0, \pi$.

(ii) $t = 0, \pi/2$
$h(3,0) = 9$
$h(0,2) = 12$

Absolute maximum $= h(0,2) = 12$ when $t = \pi/2$
Absolute minimum $= h(3,0) = 9$ when $t = 0$.

□