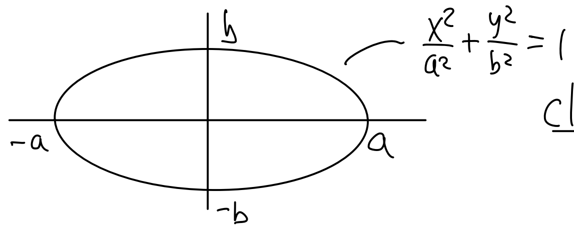
Classification of Quadratic Constraints

2-variables : $g(x,y) = Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F$

(Conic Section)

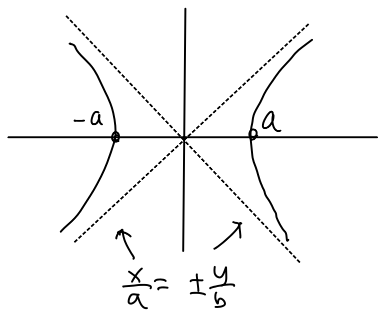
Typical examples for level curve $g(x,y) = c$

(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a, b > 0$ Ellipse (circle if $a = b$)



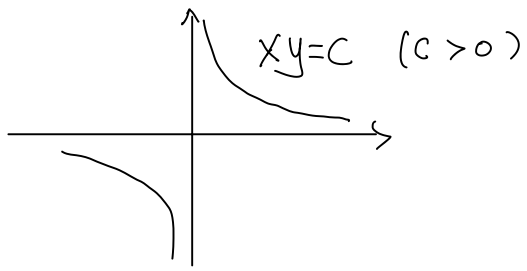
closed & bounded

(ii) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $a, b > 0$ Hyperbola



unbounded

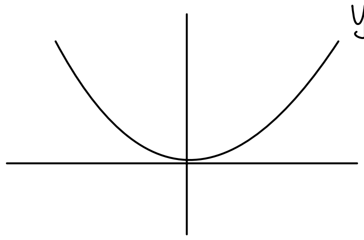
($xy = c$, $c \neq 0$, is also a hyperbola)



$$\left(x = \pm a\sqrt{1 + \frac{y^2}{b^2}} = \pm \frac{ay}{b}\sqrt{1 + \frac{b^2}{y^2}} \sim \pm \frac{a}{b} y \quad \text{as } |y| \to +\infty \right)$$

(iii) $y = ax^2$, $a \neq 0$ <u>Parabola</u>

(only 1 quadratic term)



$y = ax^2$

$a > 0$

<u>unbounded</u>

(iv) <u>Degenerate Cases</u> $(a>0, b>0)$

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0 \longrightarrow$ a point $(0,0)$

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1 \longrightarrow$ empty set

- $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0 \longrightarrow \dfrac{x}{a} = \pm \dfrac{y}{b}$ a pair of intersecting lines

  $(xy = 0)$

- $x^2 = c \longrightarrow x = \pm\sqrt{c}$
  - a pair of parallel lines if $c>0$
  - a "double" line if $c=0$
  - empty set if $c<0$

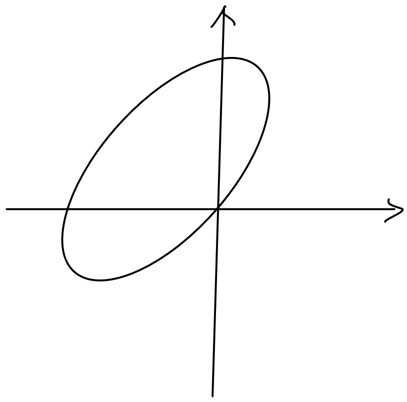
> <u>Fact</u>: By a change of coordinates, any <u>quadratic constraint</u> $g(x,y)=c$ can be transformed to one of the form above.

(<u>Proof</u>: Omitted)

$\boxed{\text{Hence \underline{level sets of quadratic constraints} are \underline{ellipse}, \underline{hyperbola}, \underline{parabola}, \& \underline{degenerated cases}.}}$

eg $17x^2 - 12xy + 8y^2 + 16\sqrt{5}x - 8\sqrt{5}y = 0$

$$\Leftrightarrow \quad \frac{u^2}{1^2} + \frac{v^2}{2^2} = 1$$



where $\begin{cases} u = \dfrac{2x-y}{\sqrt{5}} + 1, \\ v = \dfrac{x+2y}{\sqrt{5}} \end{cases}$

(Check!)

Remark: Ellipse is closed and bounded $\Rightarrow$ Any continuous $f(x,y)$ restricted to an ellipse has global max & min.

Not the case for hyperbola & parabola.