Lagrange Multipliers

(A method for *finding extrema under constraints*)

eg1 In previous example of finding global max/min of $f(x,y) = x^2+2y^2-x+3$ for $x^2+y^2 \le 1$, one need to find (in step 2) the max/min values of f on the boundary $x^2+y^2=1$.

In otherwords, finding global max/min (on $x^2+y^2=1$)

of $\quad f(x,y) = x^2+2y^2-x+3$

under constraint $\quad g(x,y) = x^2+y^2 = 1$

Another typical example:

eg2 Find the point on the parabola $x^2=4y$ closest to $(1,2)$.

i.e. Find (global) minimum of

$$f(x,y) = (x-1)^2+(y-2)^2$$

(equivalent to, but easier than $\sqrt{(x-1)^2+(y-2)^2}$)

under constraint

$$g(x,y) = x^2-4y = 0.$$

Remark: In both examples, constraints are expressed as level set $g=c$ for some constant c.

Thm (Lagrange Multipliers)

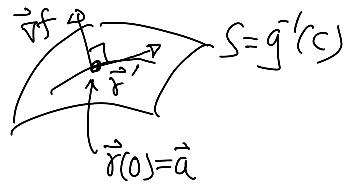
Let
- $f, g : \Omega \to \mathbb{R}$ be C^1 functions, ($\Omega \subset \mathbb{R}^n$ open)
- $S = g^{-1}(c) = \{x \in \Omega : g(x) = c\}$ be a level set of g

Suppose
- $\vec{a} \in S$ is a local extremum of f restricted to S (i.e. under the constraint $g = c$)
- $\vec{\nabla} g(\vec{a}) \neq \vec{0}$

Then
- $\vec{\nabla} f(\vec{a}) = \lambda \vec{\nabla} g(\vec{a})$ for some $\lambda \in \mathbb{R}$
- $g(\vec{a}) = c$

where λ is called a Lagrange Multiplier

(Pf: Omitted) Idea: $f(\vec{\gamma}(t))$ has an local extreme at $\vec{a}$ $\Rightarrow 0 = \left.\frac{d}{dt}\right|_{t=0} f(\vec{\gamma}(t))$



$$\Rightarrow \quad 0 = \vec{\nabla} f(\vec{a}) \cdot \vec{\gamma}'(0) \quad \text{(by Chain rule)}$$

Since it is true for all curves on S passing thro. $\vec{a}$,

$\Rightarrow$ $\vec{\nabla} f(\vec{a})$ perpendicular to $S = g^{-1}(c)$ at $\vec{a}$

$\Rightarrow$ $\vec{\nabla} f(\vec{a})$ is in the direction (or negative) of $\vec{\nabla} g(\vec{a})$.

i.e. $\vec{\nabla} f(\vec{a}) = \lambda \vec{\nabla} g(\vec{a})$ for some $\lambda \in \mathbb{R}$ #

(Cont'd) <u>Idea</u> (of the reduction to unconstrainted problem)

$$F(\vec{x}, \lambda) = F(x_1, \cdots, x_n, \lambda) \quad \text{is of } n+1 \text{ variables}$$
$$= f(\vec{x}) - \lambda(g(\vec{x}) - c)$$

critical pts :

$$\underset{\mathbb{R}^{n+1}}{\vec{0}} = \underset{(n+1)\text{-variables}}{\vec{\nabla} F} = \left(\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial \lambda} \right)$$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x_i} = \dfrac{\partial f}{\partial x_i} - \lambda \dfrac{\partial g}{\partial x_i} & \forall i = 1, \cdots, n \\ 0 = \dfrac{\partial F}{\partial \lambda} = -(g - c) \end{cases}$$

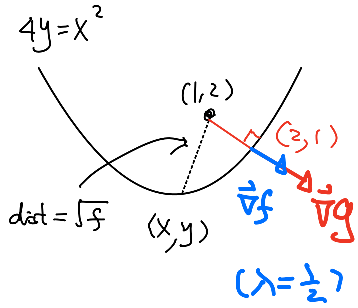
n-variables

$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial x_i} = \lambda \dfrac{\partial g}{\partial x_i} & \forall i = 1, \cdots, n \quad (\Leftrightarrow \vec{\nabla} f = \lambda \vec{\nabla} g) \\ g = c \end{cases}$$

$\Rightarrow$ extreme of f under the constraint $g = c$
by Lagrange multipliers Thm. #

eg 2 (cont'd) minimize $f(x,y) = (x-1)^2 + (y-2)^2$

under constraint $g(x,y) = x^2 - 4y = 0$



Soln: Consider

$$F(x,y,\lambda) = f(x,y) - \lambda\,(g(x,y) - 0)$$

$$= (x-1)^2 + (y-2)^2 - \lambda(x^2 - 4y)$$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x} = 2(x-1) - 2\lambda x & \text{———— (1)} \\ 0 = \dfrac{\partial F}{\partial y} = 2(y-2) + 4\lambda & \text{———— (2)} \\ 0 = \dfrac{\partial F}{\partial \lambda} = -(x^2 - 4y) & \text{———— (3)} \end{cases}$$

(2) $\Rightarrow \quad 2\lambda = 2 - y$

Put into (1) $\Rightarrow \quad 0 = 2(x-1) - (2-y)x$

$\Rightarrow \quad y = \dfrac{2}{x}$

Put into (3) $\Rightarrow \quad x^2 - \dfrac{8}{x} = 0 \Rightarrow x^3 = 8 \Rightarrow x = 2$

hence $y = 1$

$\therefore$ (2,1) is the only critical point.

$\Rightarrow$ f has a minimum at $(2,1) \in g^{-1}(0)$

with value $f(2,1) = (2-1)^2 + (1-2)^2 = 2$

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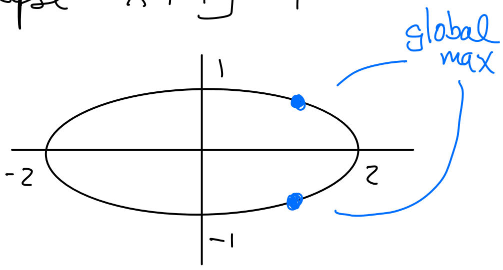
eg3 Maximize $f(x,y) = xy^2$ on the ellipse $x^2+4y^2=4$

Soln :

$$\begin{cases} f(x,y) = xy^2 \\ g(x,y) = x^2+4y^2 \end{cases}$$



& hence consider

$$F(x,y,\lambda) = xy^2 - \lambda(x^2+4y^2-4)$$

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = y^2 - 2\lambda x \\ 0 = \frac{\partial F}{\partial y} = 2xy - 8\lambda y \\ 0 = \frac{\partial F}{\partial \lambda} = 4 - x^2 - 4y^2 \end{cases}$$

(Ex!) By "simple" calculation, we have

$$(x,y) = (\pm 2, 0) \quad \text{or} \quad \left(\pm\sqrt{\tfrac{4}{3}}, \pm\sqrt{\tfrac{2}{3}}\right)$$

$$\left(= \begin{matrix} (2,0) \\ (-2,0) \end{matrix} \quad \& \quad \begin{matrix} \left(\frac{2}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) & \left(\frac{2}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right) \\ \left(-\frac{2}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) & \left(-\frac{2}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right) \end{matrix} \right)$$

are all critical points of the problem.

Comparing values of f at all these 6 critical pts:

$$f(\pm 2, 0) = 0$$

$$f\left(\frac{2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = \frac{2}{\sqrt{3}} \cdot \frac{2}{3} = \frac{4}{3\sqrt{3}} \quad \longleftarrow \text{max}$$

$$f\left(-\frac{2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = -\frac{2}{\sqrt{3}} \cdot \frac{2}{3} = -\frac{4}{3\sqrt{3}} \quad \longleftarrow \text{min}$$

$\therefore$ For $f(x,y)$ on $g(x,y)=4$, the

global max value $= \frac{4}{3\sqrt{3}}$ at $\left(\frac{2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$

(global min value $= -\frac{4}{3\sqrt{3}}$ at $\left(-\frac{2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$)

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eg 1 (cont'd) Using Lagrange multiplier, find global max/min of

$$f(x,y) = x^2+2y^2-x+3 \quad \text{on} \quad x^2+y^2=1$$

(Step 2 of the original global max/min problem on $x^2+y^2 \le 1$)

Soln: Let $\begin{cases} f(x,y) = x^2+2y^2-x+3 \\ g(x,y) = x^2+y^2 \end{cases}$

& $F(x,y,\lambda) = x^2+2y^2-x+3 - \lambda(x^2+y^2-1)$

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = 2x-1-2\lambda x & \text{——— (1)} \\ 0 = \frac{\partial F}{\partial y} = 4y - 2\lambda y & \text{——— (2)} \\ 0 = \frac{\partial F}{\partial \lambda} = -(x^2+y^2-1) & \text{——— (3)} \end{cases}$$

(2) $\Rightarrow$ $(2-\lambda)y = 0$

$\Rightarrow$ $y=0$ or $\lambda = 2$

If $y=0$, then (3) $\Rightarrow$ $x^2=1$ $\Rightarrow$ $x=\pm 1$

Hence $(x,y) = (\pm 1, 0)$ are critical pts.

If $y \neq 0$, then $\lambda = 2$

$\therefore$ (1) $\Rightarrow$ $\quad 2x - 1 - 2(2)x = 0$

$\Rightarrow \quad x = -\frac{1}{2}$

Put into (3) $\quad (\frac{1}{2})^2 + y^2 = 1 \Rightarrow y^2 = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$

$\therefore \quad (x,y) = (-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ are critical pts.

All together, the critical pts. are

$$(x,y) = (\pm 1, 0),\ (-\tfrac{1}{2}, \pm\tfrac{\sqrt{3}}{2})$$

Comparing values $\quad f(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = \frac{21}{4} \leftarrow$ max (on $x^2+y^2=1$)

$f(1, 0) = 3 \longleftarrow$ min (on $x^2+y^2=1$)

$f(-1, 0) = 5$

$\therefore$ Global max. of f on $\{x^2+y^2=1\} = \frac{21}{4}$ at $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$

Global min. of f on $\{x^2+y^2=1\} = 3$ at $(1, 0)$

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