## Matrix form for $2^{nd}$ order Taylor Polynomial

**Def:** Let $f: \Omega \to \mathbb{R}$ be $C^2$ ($\Omega \subseteq \mathbb{R}^n$, open).
Then the Hessian matrix of $f$ at $\vec{a} \in \Omega$ is

$$Hf(\vec{a}) = \begin{bmatrix} f_{x_1x_1}(\vec{a}) & \cdots & f_{x_1x_n}(\vec{a}) \\ \vdots & & \vdots \\ f_{x_nx_1}(\vec{a}) & \cdots & f_{x_nx_n}(\vec{a}) \end{bmatrix}$$

Remarks (1) $Hf(\vec{a})$ is $n \times n$ symmetric (by Clairaut's Thm)

(2) In Textbook, Hessian of $f$ = $\det(Hf(\vec{a}))$

So we emphasize our definition is a matrix (More common in advanced level math)

eg: $f(x,y)$ at $(0,0)$

$$Hf(0,0) = \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} \qquad (f_{xy} = f_{yx})$$

$$[x, y] \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \underbrace{f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2}_{2^{nd} \text{ order term in Taylor polynomials (up to a factor } \frac{1}{2!})}$$

## $2^{nd}$ order Taylor polynomial of $f$ at $\vec{a}$ in matrix form

$$\boxed{P_2(\vec{x}) = f(\vec{a}) + \vec{\nabla} f(\vec{a})(\vec{x}-\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a})^T Hf(\vec{a})(\vec{x}-\vec{a})}$$

where $\vec{\nabla} f(\vec{a})$ regarded as row vector $[f_{x_1}(\vec{a}) \cdots f_{x_n}(\vec{a})]$,

$\vec{x}-\vec{a}$ regarded as column vector $\begin{bmatrix} x_1-a_1 \\ \vdots \\ x_n-a_n \end{bmatrix}$

& $(\vec{x}-\vec{a})^T$ is the transpose $[x_1-a_1 \ \cdots \ x_n-a_n]$ (row vector)

eg $g(x,y) = \dfrac{\ln x}{1-y}$. Find $P_2(x,y)$ at $(1,0)$ using matrix form.

Soln: $g(1,0) = 0$

$$\vec{\nabla} g = \left[\frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial y}\right] = \left[\frac{1}{x(1-y)} \quad \frac{\ln x}{(1-y)^2}\right]$$

$$Hg = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix} = \begin{bmatrix} -\frac{1}{x^2(1-y)} & \frac{1}{x(1-y)^2} \\ \frac{1}{x(1-y)^2} & \frac{\ln x}{(1-y)^3} \end{bmatrix}$$

$$\Rightarrow \vec{\nabla} g(1,0) = [1, 0], \quad Hg(1,0) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\therefore P_2(x,y) = g(1,0) + \vec{\nabla} g(1,0)\begin{bmatrix} x-1 \\ y \end{bmatrix} + \frac{1}{2}[x-1 \ \ y]\, Hg(1,0) \begin{bmatrix} x-1 \\ y \end{bmatrix}$$

$$= 0 + [1 \ \ 0]\begin{bmatrix} x-1 \\ y \end{bmatrix} + \frac{1}{2}[x-1 \ \ y]\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y \end{bmatrix}$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + (x-1)y \quad \text{(Check!)} \quad ※$$

## Application to local max/min

If $f$ is $C^2$, and $\vec{a}$ is a critical point of $f$.

Then $\vec{\nabla} f(\vec{a}) = \vec{0}$

$$\Rightarrow \quad f(\vec{x}) \simeq P_2(\vec{x}) = f(\vec{a}) + \overset{\vec{0}}{\cancel{\vec{\nabla} f(a)}}(\vec{x}-\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a})^T Hf(a)(\vec{x}-\vec{a})$$

$$f(\vec{x}) - f(\vec{a}) \simeq \frac{1}{2}(\vec{x}-\vec{a})^T Hf(a)(\vec{x}-\vec{a})$$

$\therefore$ If $(\vec{x}-\vec{a})^T Hf(a)(\vec{x}-\vec{a}) < 0 \quad \forall \vec{x}$ near $\vec{a}$

then $f(\vec{x}) < f(\vec{a}) \quad \forall \vec{x}$ near $\vec{a}$

$\Rightarrow \quad \vec{a}$ is a <u>local max</u>

If $(\vec{x}-\vec{a})^T Hf(a)(\vec{x}-\vec{a}) > 0 \quad \forall \vec{x}$ near $\vec{a}$

then $f(\vec{x}) > f(\vec{a}) \quad \forall \vec{x}$ near $\vec{a}$

$\Rightarrow \quad \vec{a}$ is a <u>local min</u>

So we need to study when is a sym matrix $H$ satisfies

$$\vec{v}^T H \vec{v} > 0 \quad \forall \text{ vector } \vec{v} \neq \vec{0}$$

and

$$\vec{v}^T H \vec{v} < 0 \quad \forall \text{ vector } \vec{v} \neq \vec{0}$$

Hence we make the following