Summary: Jacobian Matrix

(1) 1-variable, real-valued: $f : \Omega \subseteq \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$

$$Df(x) = \frac{df}{dx} \quad (1\times 1 \text{ matrix, a scalar})$$

(2) Multivariable, real-valued $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, $\vec{x} \mapsto f(\vec{x})$

$$Df(\vec{x}) = \vec{\nabla} f(\vec{x}) = \left(\frac{\partial f}{\partial x_1}(\vec{x}), \cdots, \frac{\partial f}{\partial x_n}(\vec{x})\right)$$

($1\times n$ matrix, (row) vector in $\mathbb{R}^n$)

(3) Multivariable, vector-valued $\vec{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto \vec{f}(\vec{x})$

$$D\vec{f}(\vec{x}) = \begin{bmatrix} - \vec{\nabla} f_1 - \\ \vdots \\ - \vec{\nabla} f_n - \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \quad (m\times n \text{ matrix})$$

(4) 1-variable, vector-valued $\vec{\gamma} : I \subseteq \mathbb{R} \to \mathbb{R}^m$
(a curve in $\mathbb{R}^n$) $t \mapsto \vec{\gamma}(t) = (x_1(t), \cdots, x_m(t))$

$$D\vec{\gamma}(t) = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_m}{dt} \end{bmatrix} \quad (m\times 1 \text{ matric, column vector in } \mathbb{R}^m)$$

Chain Rule in classical notation

$$(x_1, \cdots, x_k) \longrightarrow (y_1, \cdots, y_n) \longrightarrow (g_1, \cdots, g_m)$$

$$\begin{cases} g_i = g_i(y_1, \cdots, y_n) & \text{are functions of } y_1, \cdots, y_n \\ y_j = y_j(x_1, \cdots, x_k) & \text{are functions of } x_1, \cdots, x_k \end{cases}$$

we can regard $g_i = g_i(x_1, \cdots, x_k)$ as functions of $x_1, \cdots, x_k$

abuse of notation

Then the ij-entry of the Chain rule

$$\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_k} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_k} \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial y_1} & \cdots & \frac{\partial g_m}{\partial y_n} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_k} \\ \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_k} \end{bmatrix}$$

is

$$\boxed{\begin{aligned} \frac{\partial g_i}{\partial x_j} &= \frac{\partial g_i}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_j} + \cdots + \frac{\partial g_i}{\partial y_n} \cdot \frac{\partial y_n}{\partial x_j} \\ &\left(= \sum_{l=1}^{n} \frac{\partial g_i}{\partial y_l} \cdot \frac{\partial y_l}{\partial x_j} \right) \end{aligned}}$$

Application of Chain Rule

Level Set

Recall: $f: \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$

$$\begin{array}{ccc} \vec{x} & \longrightarrow & c \in \mathbb{R} \\ & & \| \\ & & f(\vec{x}) \end{array}$$

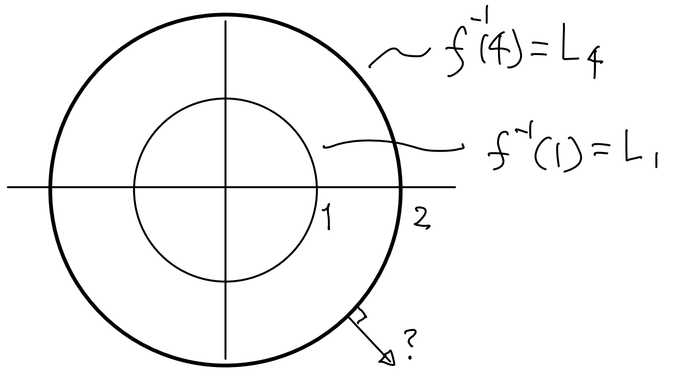
$L_c = f^{-1}(c) = \{ \vec{x} \in \Omega : f(\vec{x}) = c \}$.

$\uparrow$ Level set of f (at level c).

eg: $f(x,y) = x^2 + y^2$

$f^{-1}(1) = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$

$f^{-1}(4) = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 4 \}$



Thm Let
- $f: \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$, open)
- $c \in \mathbb{R}$
- $\vec{a} \in S = f^{-1}(c)$ (S = a level set of f)

Suppose
- f is <u>differentiable</u> at $\vec{a}$,
- $\vec{\nabla} f(\vec{a}) \neq \vec{0}$

Then $\vec{\nabla} f(\vec{a}) \perp S$ at $\vec{a}$

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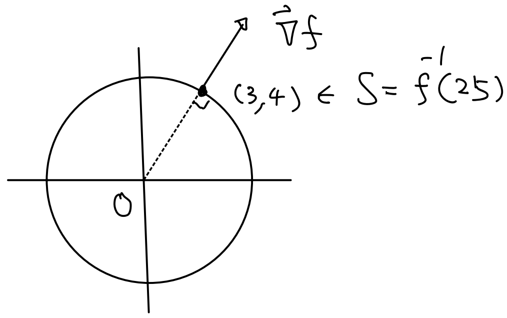
eg 1 $f(x,y) = x^2+y^2$

$\vec{\nabla} f = (2x, 2y)$

At $(3,4) \in S = f^{-1}(25)$

(i.e. level $c = 25$)



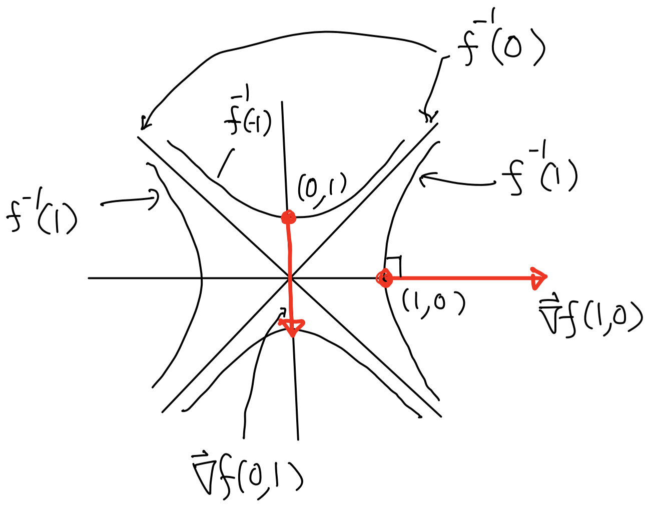
$\vec{\nabla} f(3,4) = (6,8) \perp S = f^{-1}(25)$ at the point $(3,4)$

eg 2 $f(x,y) = x^2-y^2$

$\vec{\nabla} f = (2x, -2y)$

$\vec{\nabla} f(1,0) = (2,0)$

$\vec{\nabla} f(0,1) = (0,-2)$

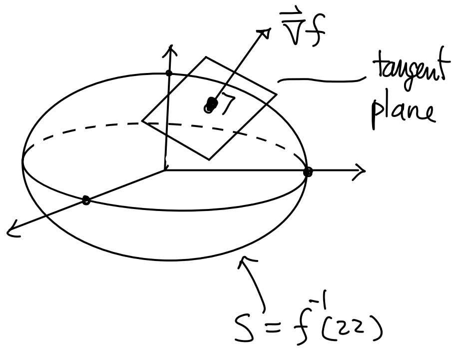


(Try other points)

(What happen at $(0,0) \in f^{-1}(0)$?)

eg 3 $S: x^2+4y^2+9z^2 = 22$ (Ellipsoid)

Find equation of tangent plane of S at the point $(3,1,1)$



(Check: $(3,1,1)$ is on S)

Soln: Let $f(x,y,z) = x^2+4y^2+9z^2$

Then $S = f^{-1}(22)$

$\vec{\nabla} f = (2x, 8y, 18z)$

$\Rightarrow \quad \vec{\nabla} f(3,1,1) = (6,8,18) \perp S$ at $(3,1,1)$

i.e. $\vec{\nabla} f(3,1,1)$ is a normal to the tangent plane at $(3,1,1)$

$$\Rightarrow \quad \vec{\nabla} f(3,1,1) \cdot (x-3, y-1, z-1) = 0$$

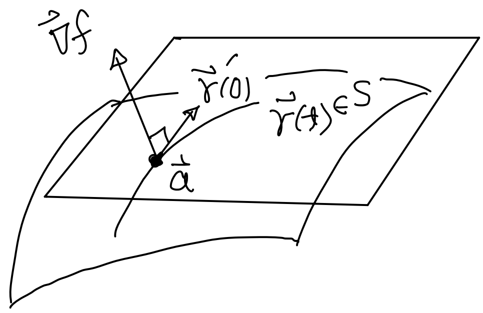
is the equation of the tangent plane.

$$\therefore \quad 6(x-3) + 8(y-1) + 18(z-1) = 0$$

or $\quad 3x + 4y + 9z = 22$

is the required equation of the tangent plane at $(3,1,1)$. ※

Proof of the Thm: $(\vec{\nabla} f \perp S)$



Let $\vec{\gamma}(t)$ be a curve on S passing through the point $\vec{a}$ such that $\vec{\gamma}(0) = \vec{a}$

Then $f(\vec{\gamma}(t)) = c, \ \forall t \quad$ (because $\vec{\gamma}(t) \in S = f^{-1}(c)$)

Chain rule $\Rightarrow \quad 0 = \left.\frac{d}{dt}\right|_{t=0} \left(f(\vec{\gamma}(t)) \right) = \vec{\nabla} f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t) \Big|_{t=0}$

$$= \vec{\nabla} f(\vec{a}) \cdot \underbrace{\vec{\gamma}'(0)}$$

tangent vector of $\vec{\gamma}(t)$ which is also a tangent vector to S.

$\vec{\nabla} f(\vec{a}) \perp$ all curves on S at $\vec{a}$

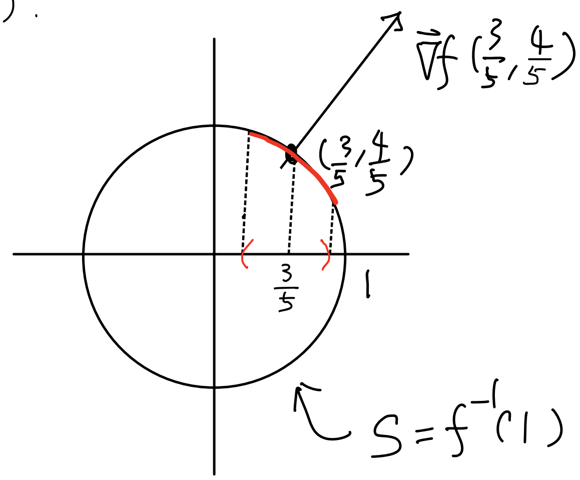
$\Rightarrow \vec{\nabla} f(\vec{a}) \perp S$ at $\vec{a}$ ※

Another Application of Chain Rule:

Implicit Differentiation

eg 1 $C: x^2+y^2=1$ (y can be solved in term of x (for most x))

Find $\frac{dy}{dx}$ at $(\frac{3}{5}, \frac{4}{5})$.



Soln: $x^2 + (y(x))^2 = 1$

near $(x,y) = (\frac{3}{5}, \frac{4}{5})$

$$\Rightarrow \quad \frac{d}{dx}(x^2+(y(x))^2) = 1$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y} \quad (\text{provided } y \neq 0)$$

At the point $(\frac{3}{5}, \frac{4}{5}) \Rightarrow \frac{dy}{dx} = -\frac{3}{4}$

Remark: One cannot solve y as a function of x near the points $(1,0)$ and $(-1,0)$ which correspond to "$y=0$".

eg 2 Consider $S: x^3+z^2+ye^{xz}+z\cos y=0$

Given that z can be regarded as a function $z=z(x,y)$ of (independent) variables x,y locally near the point $(0,0,0)$. (Clearly $(0,0,0)\in S$)

Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(0,0,0)$

Soln: $\frac{\partial}{\partial x}(x^3+z^2+ye^{xz}+z\cos y)=0$

$$\Rightarrow \quad 3x^2+2z\frac{\partial z}{\partial x}+ye^{xz}\frac{\partial}{\partial x}(xz)+\frac{\partial z}{\partial x}\cos y=0$$

$$\Rightarrow \quad 3x^2+2z\frac{\partial z}{\partial x}+ye^{xz}\left(z+x\frac{\partial z}{\partial x}\right)+\frac{\partial z}{\partial x}\cos y=0$$

$$\Rightarrow \quad (3x^2+yze^{xz})+(2z+xye^{xz}+\cos y)\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x}=-\frac{3x^2+yze^{xz}}{2z+xye^{xz}+\cos y}$$

(provided $2z+xye^{xz}+\cos y\neq 0$)

$$\Rightarrow \quad \frac{\partial z}{\partial x}(0,0)=0$$

Similarly, $\frac{\partial}{\partial y}(x^3+z^2+ye^{xz}+z\cos y)=0$

(check!) $\Rightarrow$ $$\frac{\partial z}{\partial y}=\frac{z\sin y-e^{xz}}{2z+xye^{xz}+\cos y}$$

(provided $2z+xye^{xz}+\cos y\neq 0$)

$$\Rightarrow \quad \frac{\partial z}{\partial y}(0,0)=-1 \quad ※$$