Total Differential (of real-valued function)

$f: \Omega \to \mathbb{R}$ ($\Omega \subseteq \mathbb{R}^n$, open) differentiable at $\vec{a} \in \Omega$.

Then linearization at $\vec{a}$:

$$f(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) + \varepsilon(\vec{x})$$

Usually denote: $\Delta f = f(\vec{x}) - f(\vec{a})$

$\Delta x_i = x_i - a_i$

Then $\Delta f \simeq \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a}) \Delta x_i$ $\left(\text{provided } \lim_{\vec{x} \to \vec{a}} \frac{|\varepsilon(\vec{x})|}{\|\vec{x} - \vec{a}\|} = 0\right)$

Classically, this approximation is presented as

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})\, dx_i \quad \left(\text{thinking: } \text{"}\Delta f \to df\text{"}, \ \text{"}\Delta x_i \to dx_i\text{"}\right)$$

Def: Let $\begin{cases} \bullet\ f: \Omega \to \mathbb{R}, \ (\Omega \subseteq \mathbb{R}^n, \text{ open}) \\ \bullet\ \vec{a} \in \Omega \end{cases}$

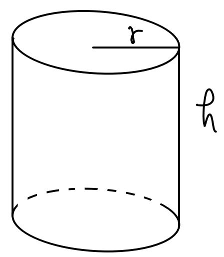
Suppose that f is <u>differentiable</u> on Ω. Then the <u>total differential</u> of f at $\vec{a}$ is defined to be the (formal) expression:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})\, dx_i$$

<u>Remark</u>: In the future, df and dx_i can be interpreted as a linear maps from $\mathbb{R}^n$ to $\mathbb{R}$.

eg: Let $V(r,h) = \pi r^2 h$

(Volume of the Cylinder)



V is differentiable (because V is C^1)

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$= 2\pi r h \, dr + \pi r^2 dh$$

For application:

Suppose we want to approximate the change of V

when (r,h) changes $(r,h) = (3, 12)$ to $(3+0.08, 12-0.3)$

Then let $dr = \Delta r = 0.08$

$dh = \Delta h = -0.3$,

we have

$$\Delta V \simeq dV = 2\pi r h \, dr + \pi r^2 dh$$

$$= 2\pi \cdot 3 \cdot 12 \cdot 0.08 + \pi 3^2 \cdot (-0.3)$$

$$= 3.06\pi \simeq 9.61$$

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Properties of Total Differential

If
- $f, g : \Omega \to \mathbb{R}$ ($\Omega \subseteq \mathbb{R}^n$, open) are differentiable, and
- $c \in \mathbb{R}$ is a constant.

Then

(1) $d(f \pm g) = df \pm dg$,

(2) $d(cf) = c\,df$

(3) $d(fg) = g\,df + f\,dg$

(4) $d\left(\frac{f}{g}\right) = \dfrac{g\,df - f\,dg}{g^2}$ provided $g \neq 0$

(Pf: Easily from properties of partial derivatives)

Summary (on differentiation of a real-valued function on $\mathbb{R}^n$)

$f: \mathbb{R}^n \to \mathbb{R}$

A. Types of differentiations (derivatives)

- Directional Derivative :

$$D_{\vec{u}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a}+t\vec{u}) - f(\vec{a})}{t} \qquad (\|\vec{u}\| = 1)$$

- Partial derivatives :

$$\frac{\partial f}{\partial x_i}(\vec{a}) = D_{\vec{e}_i} f(\vec{a}), \quad \vec{e}_i = (0, \cdots, \underset{\uparrow \text{ ith component}}{1}, \cdots 0)$$

- Gradient

$$\vec{\nabla} f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \cdots, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$$

- Total Differential

$$df(\vec{a}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})\, dx_i$$

- Higher Derivatives

$$\frac{\partial^{k_1 + \cdots + k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}(\vec{a})$$

(provided f is C^k, $k = k_1 + \cdots + k_n$)

$\uparrow$ all partial derivatives up to order k exist & cts.

B. Linear approximation

- $L(\vec{x}) = f(\vec{a}) + \vec{\nabla} f(\vec{a}) \cdot (\vec{x}-\vec{a})$

- $f(\vec{x}) = L(\vec{x}) + \varepsilon(\vec{x})$

  $\qquad\qquad\qquad\qquad \leftarrow$ error term

- f is differentiable at $\vec{a}$

$$\Longleftrightarrow \quad \lim_{\vec{x}\to\vec{a}} \frac{|\varepsilon(\vec{x})|}{\|\vec{x}-\vec{a}\|} = 0$$

In this case, $df \simeq \Delta f$ (by identifying $dx_i = \Delta x_i$)

C. Relations among various concepts

- $C^\infty \Rightarrow \cdots \Rightarrow C^{k+1} \Rightarrow C^k \Rightarrow \cdots \Rightarrow C^1 \Rightarrow C^0$ (No reverse implication)

- f is $\underline{C^1 \text{ on an open set}}$ containing $\vec{a}$

$$\Downarrow \; \not\Uparrow$$

f is $\underline{\text{differentiable}}$ at $\vec{a}$

$$\swarrow \not\nearrow \qquad\qquad \searrow \not\nwarrow$$

$D_{\vec{u}} f(\vec{a})$ exists $\forall \vec{u} \in \mathbb{R}^n$, $\|\vec{u}\|=1$ $\quad \not\Rightarrow \;\; \not\Leftarrow \quad$ f is continuous at $\vec{a}$

$$\searrow \not\nwarrow \qquad\qquad \not\swarrow \not\nearrow$$

$\frac{\partial f}{\partial x_i}(\vec{a})$ exists, $\forall i = 1, \dots, n$

Counter examples:

eg1: $f: \mathbb{R} \to \mathbb{R}$ (in MATH2050)

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

f is differentiable on $\mathbb{R}$ but (check!)

$f'(x)$ is not continuous at $x=0$

(For $x \neq 0$, $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ limit DNE as $x \to 0$)

Similarly $g(x) = x^{2k-2} f(x)$ is k-time differentiable but $g^{(k)}(x)$ is not continuous at $x=0$ (Pf: Omitted)

Hence k-time differentiable $\not\Rightarrow C^k$.

(For multi-variable: $h(\vec{x}) = h(x_1, \cdots, x_n) = g(x_1)$)

eg2
$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x^2+y^2 = 0 \end{cases}$$

$D_{\vec{u}} f(0,0)$ exists, $\forall$ unit vector $\vec{u} = (\cos\theta, \sin\theta) \in \mathbb{R}^2$

but f is not continuous at $(0,0)$ (check!)

eg3: $f(x,y) = |x+y|$ is continuous on $\mathbb{R}^2$, but

$f_x(0,0)$, $f_y(0,0)$ DNE (check!)

eg 4 : $f(x,y) = \sqrt{|xy|}$

$f_x(0,0)$, $f_y(0,0)$ exist (in fact $=0$)

but $D_{\vec{u}} f(0,0)$ DNE for $\vec{u} \neq \pm\vec{e}_1, \pm\vec{e}_2$