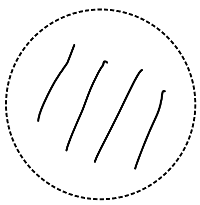
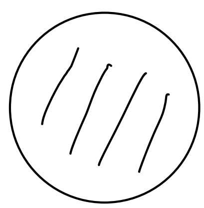
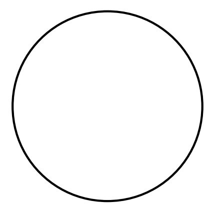
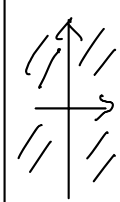
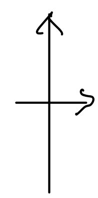
Eg:

Subset $S \subset \mathbb{R}^2$	$B_1(0,0) = \{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}$	$\overline{B_1(0,0)} = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 1\}$	$S^1 = \{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$	$\mathbb{R}^2$	ϕ (empty set)
Int(S)	$B_1(0,0)$	$B_1(0,0)$	$\emptyset$	$\mathbb{R}^2$	$\emptyset$
Ext(S)	$\mathbb{R}^2 \setminus \overline{B_1(0,0)}$ $= \{x^2+y^2>1\}$	$\mathbb{R}^2 \setminus \overline{B_1(0,0)}$	$\mathbb{R}^2 \setminus S^1$	$\emptyset$	$\mathbb{R}^2$
∂S	S^1	S^1	S^1	$\emptyset$	$\emptyset$
Open?	Yes	No	No	Yes	Yes
Closed?	No	Yes	Yes	Yes	Yes
Picture					

Remarks: (1) There are exactly two subsets of $\mathbb{R}^n$ which are <u>both</u> open and closed: $\mathbb{R}^n$ and ϕ.

(2) Some subsets of $\mathbb{R}^n$ are <u>neither</u> open <u>nor</u> closed. (eg: last lecture)

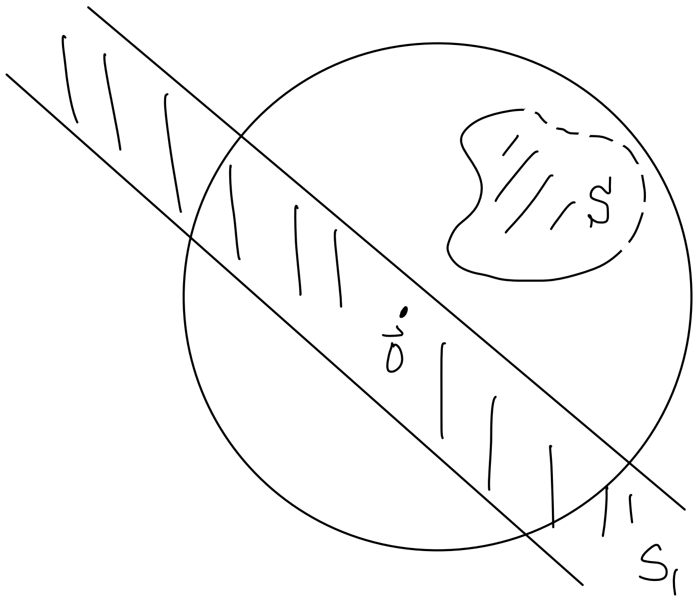
(3) For any $S \subseteq \mathbb{R}^n$, $\text{Int}(S)$ & $\text{Ext}(S)$ are <u>open</u> in $\mathbb{R}^n$;
∂S is <u>closed</u> in $\mathbb{R}^n$

(Ex: What about $\text{Int}(S) \cup \partial S$?)

Def: $S \subseteq \mathbb{R}^n$ is called <u>bounded</u> if $\exists M > 0$ such that

$$S \subseteq B_M(\vec{0}) = \{ \bar{x} \in \mathbb{R}^n : \|\vec{x}\| < M \}$$

S is called <u>unbounded</u> if it is not bounded



S is <u>bounded</u>

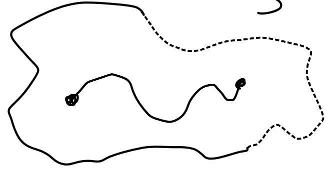
S_1 is <u>unbounded</u>

eg: $y\text{-axis} = \{ (x,y) \in \mathbb{R}^2 : x = 0 \}$ is unbounded

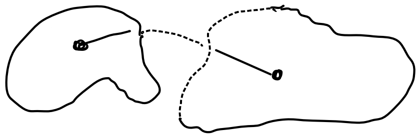
(Pf: $\forall M > 0$, $\exists (0, 2M) \in y\text{-axis}$ s.t. $(0, 2M) \notin B_M(\vec{0})$.)

Def $S \subset \mathbb{R}^n$ is called <u>path-connected</u> if any two points in S can be connected by a curve in S.

path-connected

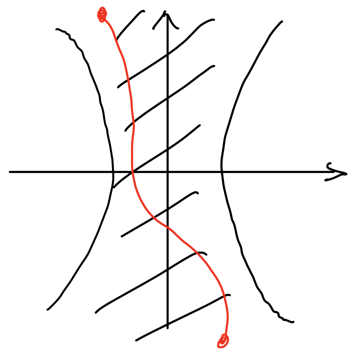


cannot joint by a curve completely inside S.

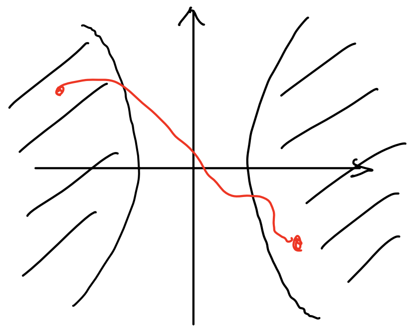


eg: $S = \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 \leq 1\}$ is path-connected

$S_1 = \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 \geq 1\}$ is not path-connected.



S



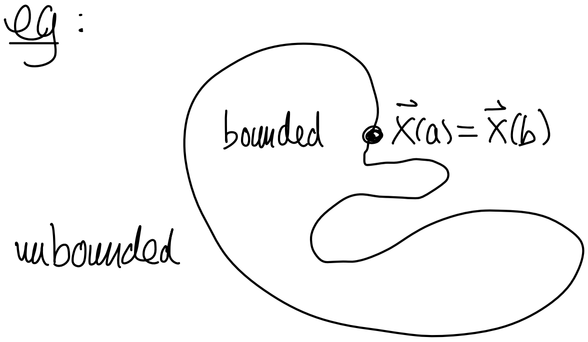
S_1

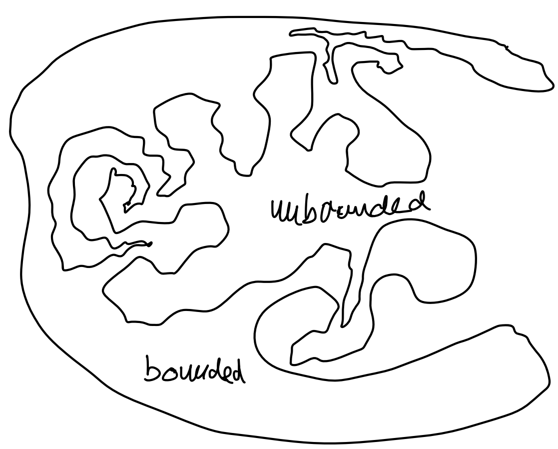
Remark: In topology, there is a different notion called "connected". We'll not discuss it.

> **Thm (Jordan Curve Theorem)**
>
> A simple closed curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into 2 path-connected components, with one bounded and one unbounded

Remark: "closed curve" means continuous curve $\vec{X}(t)$, $a \leq t \leq b$ with $\vec{X}(a) = \vec{X}(b)$. And one can show that it is a "closed subset" in $\mathbb{R}^2$.

eg:



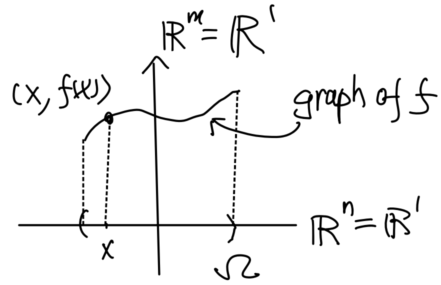


Vector-valued functions of Multivariables

$$\vec{f} : \Omega^{\subset \mathbb{R}^n} \longrightarrow \mathbb{R}^m \qquad \text{How to visualize it ?}$$

(1) Graph of $\vec{f}$

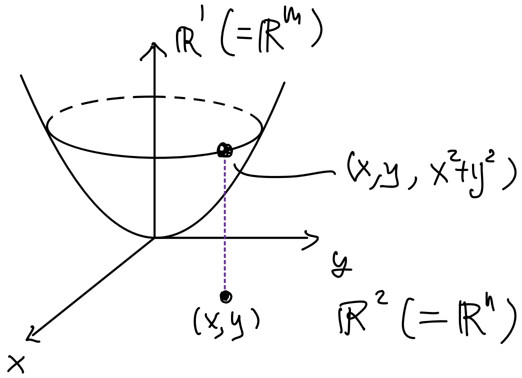
$n = m = 1$



$$\text{Graph}(\vec{f}) = \{ (\underset{\mathbb{R}^n}{\underset{\cap}{\vec{x}}}, \underset{\mathbb{R}^m}{\underset{\cap}{\vec{f}(\vec{x})}}) : \vec{x} \in \Omega \}$$

$$\subseteq \mathbb{R}^{n+m}$$

eg : $n = 2,\ m = 1$: $g : \mathbb{R}^2 \to \mathbb{R}$ defined $g(x,y) = x^2 + y^2$



graph (g) is the surface

$$= \{ (x, y, x^2+y^2) \in \mathbb{R}^3 : (x,y) \in \mathbb{R}^2 \} \subseteq \mathbb{R}^3$$

(In general, it is impossible to draw the graph for $n + m > 3$.)

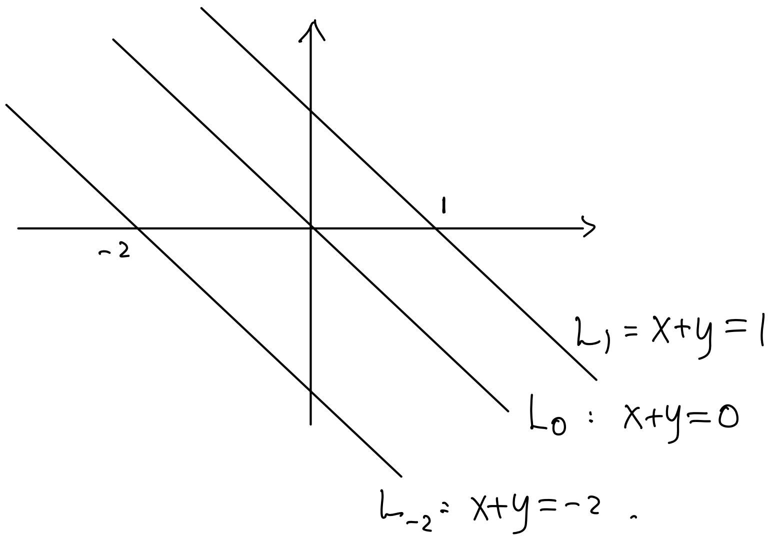
(2) Level set of $\vec{f} : \Omega^{\subset \mathbb{R}^n} \longrightarrow \mathbb{R}^m$

If $\vec{c} \in \mathbb{R}^m$, define the level set at $\vec{c}$ to be

$$L_{\vec{c}} = \{ \vec{x} \in \Omega \subset \mathbb{R}^n : \vec{f}(\vec{x}) = \vec{c} \} = (\vec{f})^{-1}(\vec{c}) \subseteq \Omega \subseteq \mathbb{R}^n .$$

eg: $f(x,y) = x+y$, $\Omega = \mathbb{R}^2$ $\quad$ ($m=1 \Rightarrow \vec{c} \in \mathbb{R}^1$, ie $\vec{c}$ is a number)

$$L_c = \{(x,y) \in \mathbb{R}^2 : f(x,y) = c\}$$
$$= \{(x,y) \in \mathbb{R}^2 : x+y = c\}$$



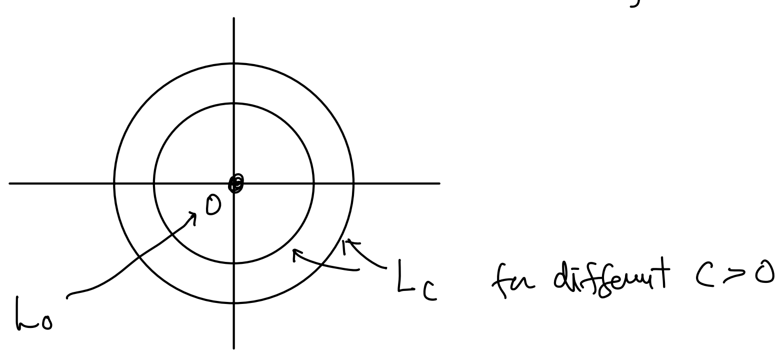
eg: $g(x,y) = x^2+y^2$, $\Omega = \mathbb{R}^2$

$$L_c = \{(x,y) \in \mathbb{R}^2 : g(x,y) = c\}$$
$$= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = c\}$$

Case 1: $c < 0$, $L_c = \emptyset$

Case 2: $c = 0$, $L_0 = \{(0,0)\}$

Case 3: $c > 0$, $L_c = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = c\}$
$= $ circle of radius $\sqrt{c}$ centered at $(0,0)$.



eg: $h(x,y) = \cos(2\pi(x^2+y^2))$, $\Omega = \mathbb{R}^2$

$L_c = \{(x,y)\in\mathbb{R}^2 : h(x,y) = c\}$

$= \{(x,y)\in\mathbb{R}^2 : \cos(2\pi(x^2+y^2)) = c\}$

Case 1: If $|c|>1$, then $L_c = \emptyset$

Case 2: If $|c|\leq 1$, then $L_c = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 = \frac{1}{2\pi}\cos^{-1}(c)\}$

subcase (a) $\cos^{-1}(c)<0$, $L_c=\emptyset$

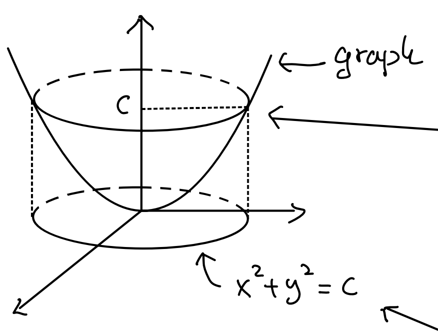
subcase (b) $\cos^{-1}(c)=0$, $L_c=\{(0,0)\}$

subcase (c) $\cos^{-1}(c)>0$, L_c = circle of radius $\sqrt{\frac{1}{2\pi}\cos^{-1}(c)}$ centered at $(0,0)$.

(depends on how you choose $\cos^{-1}(c)$, we may choose $\cos^{-1}$ s.t. subcase (a) will not occur. Further discussion omitted)

Level set $\longleftrightarrow$ graph

$g(x,y) = x^2+y^2$



cut at horizontal level

$z = c\ (\geq 0)$

$\downarrow$ project down

level set.

Limit of Multi-variable Functions

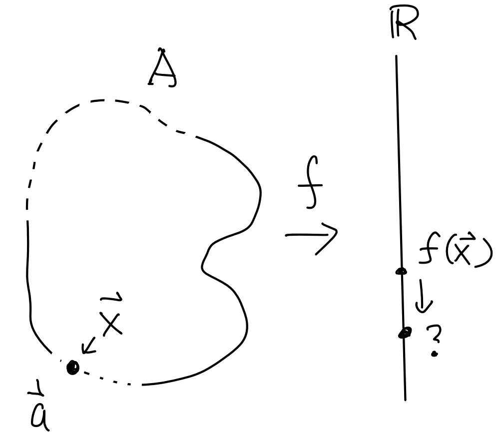
Let $A \subseteq \mathbb{R}^n$

$f: A \to \mathbb{R}$ be a function (of n-variables)

Let $\overline{A} \overset{\text{def}}{=\!=} A \cup \partial A$ the <u>closure</u> of A

For $\vec{a} \in A \cup \partial A$, we consider

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x})$$



In general n, m-dims

Def (ε-δ): Let $\vec{f}: A \subset \mathbb{R}^n \to \mathbb{R}^m$, $\vec{a} \in \overline{A} = A \cup \partial A$

We say that $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\vec{x} \in A \text{ and } 0 < \|\vec{x} - \vec{a}\| < \delta \Rightarrow \|\vec{f}(\vec{x}) - \vec{L}\| < \varepsilon$$

<u>Remarks</u> (i) $\|\vec{x} - \vec{a}\|$ = distance between $\vec{x}$ and $\vec{a}$ in $\mathbb{R}^n$

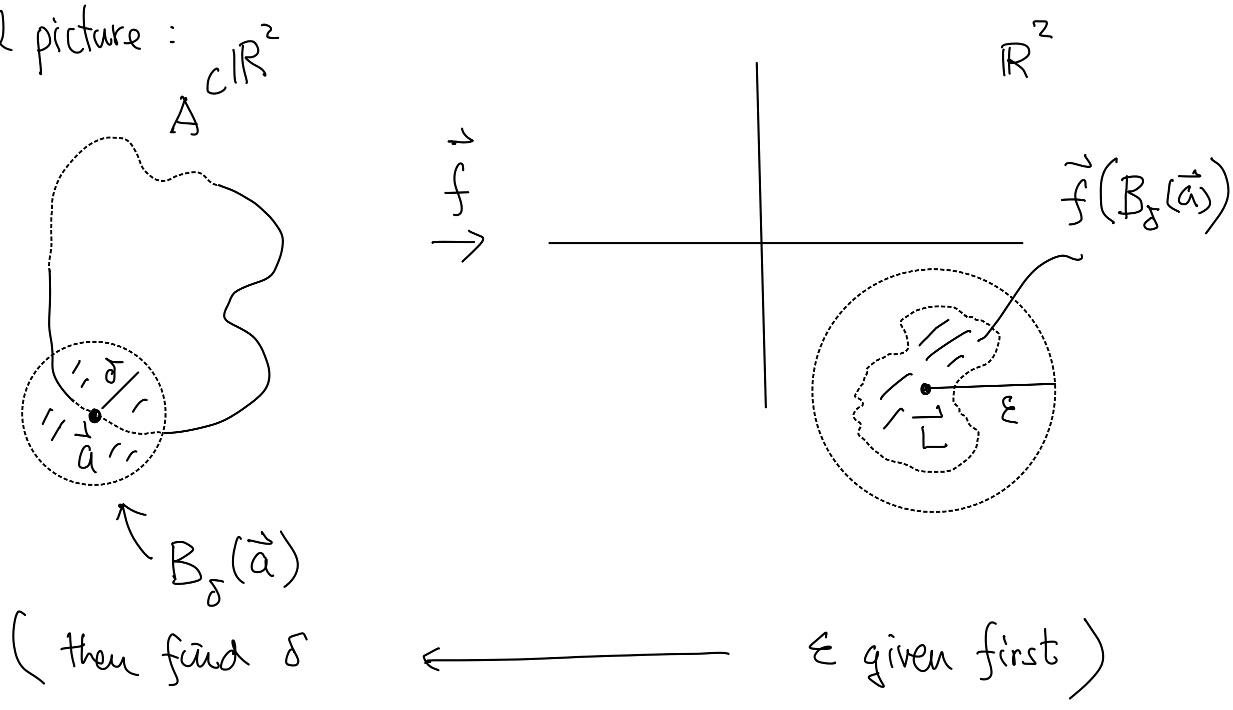
$0 < \|\vec{x} - \vec{a}\|$ means $\vec{x} \neq \vec{a}$

i.e. Considering points close to $\vec{a}$ but <u>not equal</u> to $\vec{a}$.

(ii) $\|\vec{f}(\vec{x}) - \vec{L}\|$ = distance between $\vec{f}(\vec{x})$ and $\vec{L}$ in $\mathbb{R}^m$.

If $m = 1$, $\|\vec{f}(\vec{x}) - \vec{L}\| = |f(\vec{x}) - L|$ absolute value of the different.

2 dim'l picture :



eg: $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x + y$

Illustrate that $\lim_{(x,y)\to(1,2)} f(x,y) = 3$.

Soln: [i.e. you need to show that given any $\varepsilon > 0$, one can find $\delta > 0$ such that if $0 < \|(x,y) - (1,2)\| < \delta$ then $|f(x,y) - 3| < \varepsilon$. (No need to check $(x,y) \in A$, because $A = \mathbb{R}^2$)]

Idea: $|f(x,y) - 3| = |x + y - 3|$

$= |(x-1) + (y-2)| \leq |x-1| + |y-2|$

$\|(x,y) - (1,2)\| = \sqrt{(x-1)^2 + (y-2)^2}$

For instance, for $\varepsilon = 1$, choose $\delta = \frac{1}{2}$

if $0 < \|(x,y) - (1,2)\| < \delta$, then $|x-1| \leq \|(x,y) - (1,2)\| < \frac{1}{2} (= \delta)$

$|y-2| \leq \|(x,y) - (1,2)\| < \frac{1}{2} (= \delta)$

$\Rightarrow \quad |f(x,y)-3| < \frac{1}{2}+\frac{1}{2} = 1 \ (=\varepsilon)$

Similarly, for $\varepsilon = \frac{1}{1000}$, one can choose $\delta = \frac{1}{2000}$ (Ex!)

<u>(Real) Proof:</u> For any given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{2}$. Then

$$\|(x,y)-(1,2)\| < \delta = \frac{\varepsilon}{2}$$

$$\Rightarrow |f(x,y)-3| = |x+y-3| = |(x-1)+(y-2)| \leq |x-1|+|y-2|$$
$$< \delta+\delta = \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

(Since $|x-1| \leq \|(x,y)-(1,2)\|$ & $|y-2| \leq \|(x,y)-(1,2)\|$)

$$\therefore \lim_{(x,y)\to(1,2)} f(x,y) = 3 \quad ※$$

<u>eg</u>: Let $f(x,y) = x^2+y^2$

Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ from definition.

<u>Soln</u>: Need to show that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $0 < \|(x,y)-(0,0)\| = \sqrt{x^2+y^2} < \delta$

then $\|f(x,y)-0\| = |x^2+y^2| < \varepsilon$

<u>eg</u>: $\varepsilon = \frac{1}{100}$, then $\delta = \sqrt{\varepsilon} = \frac{1}{10}$.

If $\|(x,y)-(0,0)\| < \delta = \frac{1}{10}$, then $\sqrt{x^2+y^2} < \frac{1}{10}$

$\Rightarrow \quad x^2+y^2 < \frac{1}{100}$, i.e. $\|f(x,y)-0\| < \frac{1}{100} = \varepsilon$

(Real) Proof: For any given $\varepsilon > 0$, choose $\delta = \sqrt{\varepsilon} > 0$.

Then $0 < \|(x,y) - (0,0)\| < \delta = \sqrt{\varepsilon} \Rightarrow \sqrt{x^2+y^2} < \sqrt{\varepsilon}$

$\Rightarrow x^2+y^2 < \varepsilon$, i.e. $|f(x,y) - 0| = x^2+y^2 < \varepsilon$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0$. ※

Prop: Let
- $A \subseteq \mathbb{R}^n$
- $\vec{a} \in \overline{A} = A \cup \partial A$
- $\vec{f} : A \to \mathbb{R}^m$ with

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix} = \begin{bmatrix} f_1(x_1, \cdots, x_n) \\ \vdots \\ f_m(x_1, \cdots, x_n) \end{bmatrix}$$

where $\vec{x} = (x_1, \cdots, x_n) \in A$.

Then
$$\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{L} = \begin{bmatrix} l_1 \\ \vdots \\ l_m \end{bmatrix} \Leftrightarrow \lim_{\vec{x}\to\vec{a}} f_i(\vec{x}) = l_i, \ \forall i = 1, \cdots, m.$$

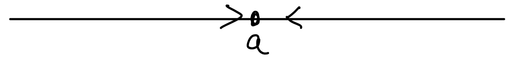
Consequence: It is good enough for us to focus on limit of real-valued function $f: A \to \mathbb{R}$ (ie. $m=1$)

eg: $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^2 : \vec{f}(x,y) = \begin{bmatrix} x+y \\ x^2+y^2+1 \end{bmatrix}$

$$\lim_{(x,y)\to(1,2)} \vec{f}(x,y) = \begin{bmatrix} \lim_{(x,y)\to(1,2)} (x+y) \\ \lim_{(x,y)\to(1,2)} (x^2+y^2+1) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad (\longleftarrow \text{Ex!})$$

Limit along a path

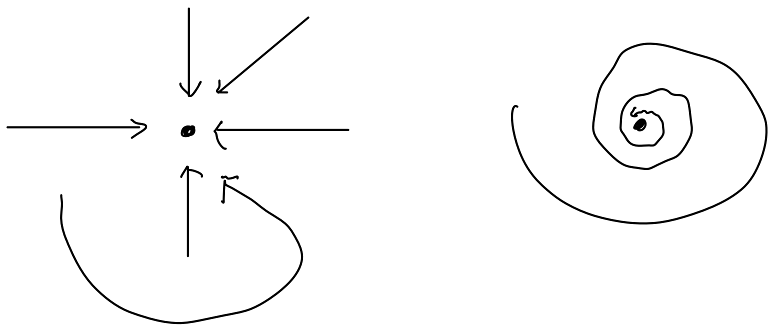
Recall: In one variable:



$$\lim_{x\to a} f(x) \text{ exists} \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

(exist & equal)

For n-variables, $n \geq 2$, there are infinitely many ways to approach a point in $\mathbb{R}^n$. Situation is very complicated.



However, we still have the following

> **Fact**: $f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, $\vec{a} \in \bar{A} = A \cup \partial A$. Then
>
> $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L \iff$ limit of $f(\vec{x})$ when $\vec{x}$ approaches to $\vec{a}$ along any curve (path) exists and equals to L

- Useful for showing limit "does not exists" (DNE) (only in our dept. not a common notation)

(i) Find a path such that the limit along that path DNE, or

(ii) Find 2 paths such that the limits along the 2 paths are different

$$\Rightarrow \lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) \text{ DNE}.$$

eg $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ $\left(\frac{x^2-y^2}{x^2+y^2} \text{ doesn't define at } (x,y)=(0,0) \right)$

Soln: (1) Along x-axis $(y=0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to0} \frac{x^2}{x^2} = 1$$

(2) Along y-axis $(x=0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to0} \frac{-y^2}{y^2} = -1$$

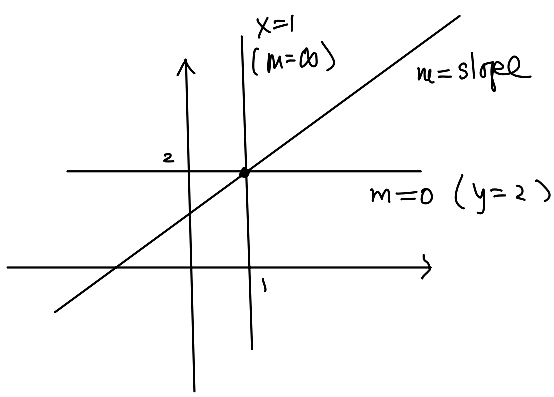
Different limits along different paths $\Rightarrow \displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ DNE.

(In fact, we can try other paths too, for instance, $y=x$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} \frac{x^2-y^2}{x^2+y^2} = 0 \qquad (\text{Ex!})$$
)

eg : Consider $\displaystyle\lim_{(x,y)\to(1,2)} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2}$ along all straight lines passing through $(1,2)$.

Soln: (1) $(m=\infty)$ Along $x=1$



$$\lim_{\substack{(x,y)\to(1,2)\\ x=1}} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2}$$

$$= \lim_{y\to2} \frac{y-2-y+2}{(y-2)^2} = 0$$

(2) ($m \neq \infty$) Along the line with slope $= m$ & passing thro. $(1,2)$

$$y-2 = m(x-1)$$

$$\lim_{\substack{(x,y)\to(1,1)\\ y-2=m(x-1)}} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2} = \lim_{\substack{(x,y)\to(1,1)\\ y-2=m(x-1)}} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2}$$

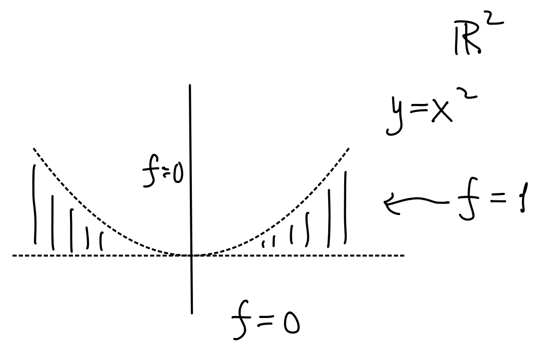
$$= \lim_{x\to 1} \frac{(x-1)\cdot m(x-1)}{(x-1)^2+(m(x-1))^2} = \frac{m}{1+m^2}$$

(Different limits for different slopes (ie different paths)

$\therefore \lim_{(x,y)\to(1,2)} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2}$ DNE.)

eg: $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 1, & \text{if } 0<y<x^2 \\ 0, & \text{otherwise} \end{cases}$$



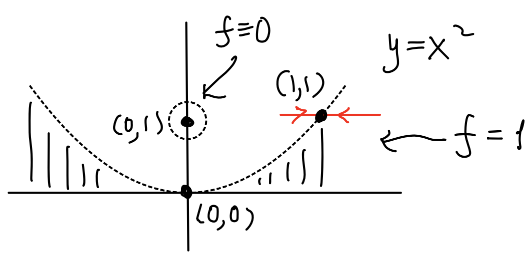
($f \equiv 1$ for |||.....|||, 0 other places)

Find $\lim_{(x,y)\to\vec{a}} f(x,y)$, where

(i) $\vec{a} = (0,1)$

(ii) $\vec{a} = (1,1)$

(iii) $\vec{a} = (0,0)$



Soln: (i) For $\vec{a} = (0,1)$, $f(x,y) = 0$ "near" $(0,1) \Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

(ii) For $\vec{a} = (1,1)$, $\lim\limits_{\substack{(x,y)\to(1,1)\\ x<1,\, y=1}} f(x,y) = \lim\limits_{\substack{x\to 1^-\\ (y=1)}} 0 = 0$

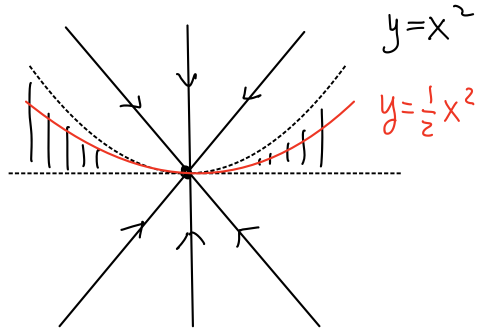
$$\lim_{\substack{(x,y)\to(1,1)\\ x>1,\, y=1}} f(x,y) = \lim_{\substack{x\to 1^+\\ (y=1)}} 1 = 1$$

Different limits for different paths $\Rightarrow \lim\limits_{(x,y)\to(1,1)} f(x,y)$ DNE

(iii) Case 1 Along y-axis ($x=0$)

$f(0,y) = 0, \ \forall y$

$$\Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) = 0$$



Case 2 Along $y = mx$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim_{x\to 0} f(x, mx) = 0 \quad \left(\begin{array}{l}\text{because } mx > x^2 \\ \text{for small } x>0\end{array}\right)$$

Case 3 Along the curve $y = \frac{1}{2}x^2$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=\frac{1}{2}x^2}} f(x,y) = \lim_{x\to 0} f(x, \tfrac{1}{2}x^2) = \lim_{x\to 0} 1 = 1$$

Case 2 & Case 3 $\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ DNE ※