13.4 Chain Rule: Two and Three Independent Variables

In Exercises 7 and 8, **(a)** express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of u and v both by using the Chain Rule and by expressing z directly in terms of u and v before differentiating. Then **(b)** evaluate $\partial z/\partial u$ and $\partial z/\partial v$ at the given point (u, v).

7. $z = 4e^x \ln y$, $x = \ln(u\cos v)$, $y = u\sin v$; $(u, v) = (2, \pi/4)$

Sol'n: a) Chain Rule:

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\ln(u\cos v) = \frac{1}{u\cos v}\cdot\cos v = \frac{1}{u}.$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\ln(u\cos v) = \frac{1}{u\cos v}\cdot -u\sin v = -\tan v.$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}u\sin v = \sin v$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}u\sin v = u\cos v.$$

So
$$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}(4e^x\ln y) = 4e^x\ln y\frac{\partial x}{\partial u} + \frac{4e^x}{y}\frac{\partial y}{\partial u}$$
$$= \frac{4e^x\ln y}{u} + \frac{4e^x\sin v}{y}$$

$$= \frac{4e^{\ln(u\cos v)}\ln(u\sin v)}{u} + \frac{4e^{\ln(u\cos v)}\cancel{\sin v}}{u\cancel{\sin v}}$$

$$= 4\cos v\ln(u\sin v) + 4\cos v$$

$$\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}\left(4e^x\ln y\right) = 4e^x\ln y\frac{\partial x}{\partial v} + \frac{4e^x}{y}\frac{\partial y}{\partial v}$$

$$= -4e^x\ln y\tan v + \frac{4e^x u\cos v}{y}$$

$$= -4e^{\ln(u\cos v)}\ln(u\sin v)\tan v + \frac{4e^{\ln(u\cos v)}\cancel{u}\cos v}{\cancel{u}\sin v}$$

$$= -4u\sin v\ln(u\sin v) + \frac{4u\cos^2 v}{\sin v}$$

Direct substitution:

$$z = 4e^x\ln y = 4e^{\ln(u\cos v)}\ln(u\sin v)$$

$$= 4u\cos v\ln(u\sin v)$$

Then $\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}\left(4u\cos v \ln(u\sin v)\right)$

$$= 4\cos v \ln(u\sin v) + \frac{4\cancel{u}\cos v}{\cancel{u\sin v}} \cdot \cancel{\sin v}$$

$$= 4\cos v \ln(u\sin v) + 4\cos v$$

$\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}\left(4u\cos v \ln(u\sin v)\right)$

$$= -4u\sin v \ln(u\sin v) + \frac{4\cancel{u}\cos v}{\cancel{u}\sin v} \cdot \cos v$$

$$= -4u\sin v \ln(u\sin v) + \frac{4\cos^2 v}{\sin v}$$

b) $\left.\frac{\partial z}{\partial u}\right|_{(2,\frac{\pi}{4})} = \left.\left(4\cos v \ln(u\sin v) + 4\cos v\right)\right|_{(2,\frac{\pi}{4})} = 4\cdot\frac{\sqrt{2}}{2}\cdot \ln\left(2\cdot\frac{\sqrt{2}}{2}\right) + 4\cdot\frac{\sqrt{2}}{2}$

$$= 2\sqrt{2}\ln(\sqrt{2}) + 2\sqrt{2}.$$

$$\left.\frac{\partial z}{\partial v}\right|_{(2,\frac{\pi}{4})} = \left.\left(-4u\sin v\,\ln(u\sin v) + \frac{4\cos^2 v}{\sin v}\right)\right|_{(2,\frac{\pi}{4})}$$

$$= -4\cdot 2\cdot\frac{\sqrt{2}}{2}\ln\left(2\cdot\frac{\sqrt{2}}{2}\right) + \frac{4\cdot\left(\frac{\sqrt{2}}{2}\right)^2}{\frac{\sqrt{2}}{2}}$$

$$= -4\sqrt{2}\ln(\sqrt{2}) + 2\sqrt{2}$$

13.4

Implicit Differentiation

Assuming that the equations in Exercises 25–30 define y as a differentiable function of x, use Theorem 8 to find the value of dy/dx at the given point.

29. $(x^3 - y^4)^6 + \ln(x^2 + y) = 1, \quad (-1, 0)$

Thm 8: Sps $F(x,y)$ differentiable and $F(x,y)=0$ defines y as a differentiable fn of x. Then, at any pt. where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Sol'n: Let $F(x,y) = (x^3-y^4)^6 + \ln(x^2+y) - 1$.

Then $$F_y\Big|_{(-1,0)} = \left(6(x^3-y^4)^5(-4y^3) + \frac{1}{x^2+y}\right)\Bigg|_{(-1,0)}$$

$$= 6((-1)^3-(0)^4)^5\cancel{(-4(0)^3)}^{\,0} + \frac{1}{(-1)^2+0} = 1 \neq 0.$$

So $\left.\frac{dy}{dx}\right|_{(-1,0)} = \left.\frac{-F_x}{F_y}\right|_{(-1,0)}$. So we compute $F_x\big|_{(-1,0)}$

$$F_x\Big|_{(-1,0)} = \left(6(x^3-y^4)^5(3x^2) + \frac{1}{x^2+y}\cdot 2x\right)\Bigg|_{(-1,0)}$$

$$= \left(18x^2(x^3-y^4)^5 + \frac{2x}{x^2+y}\right)\Bigg|_{(-1,0)} = 18\cdot\overset{1}{\cancel{(-1)^2}}\,\overset{(-1)^5}{(\cancel{(-1)^3}-(0)^4)^5} + \frac{2(-1)}{(-1)^2+0}$$

$$= -18-2 = -20$$

So $\left.\frac{dy}{dx}\right|_{(-1,0)} = \boxed{20}$.

13.4

Theory and Examples

41. Assume that $w = f(s^3 + t^2)$ and $f'(x) = e^x$. Find $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$.

Sol'n:

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial t} = f'(s^3+t^2)\cdot 2t = 2te^{s^3+t^2}$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial s} = f'(s^3+t^2)\cdot 3s^2 = 3s^2e^{s^3+t^2}$$

//.

13.4 **Differentiating Integrals** Under mild continuity restrictions, it is true that if

$$F(x) = \int_a^b g(t, x)\, dt,$$

then $F'(x) = \int_a^b g_x(t, x)\, dt$. Using this fact and the Chain Rule, we can find the derivative of

$$F(x) = \int_a^{f(x)} g(t, x)\, dt$$

by letting

$$G(u, x) = \int_a^u g(t, x)\, dt,$$

where $u = f(x)$. Find the derivatives of the functions in Exercises 59 and 60.

59. $F(x) = \int_0^{x^2} \sqrt{t^4 + x^3}\, dt$

Sol'n: Let $u = x^2$. Then $G(u,x) = \int_0^u \sqrt{t^4 + x^3}\, dt$.

Then $F(x) = G(u,x) = G(x^2, x)$. And by the chain rule we have

$$\frac{dF}{dx} = \frac{\partial G}{\partial u} \cdot \frac{\partial u}{\partial x}\bigg|_{u=x^2} + \frac{\partial G}{\partial x}\bigg|_{u=x^2}$$

$$\frac{\partial G}{\partial u}\cdot\frac{\partial u}{\partial x}\bigg|_{u=x^2} = \sqrt{x^8+x^3}\,\frac{\partial(x^2)}{\partial x} - \sqrt{0^4+x^3}\,\frac{\partial(0)}{\partial x} = 2x\sqrt{x^8+x^3}$$

derivative of antiderivative

$$\frac{\partial}{\partial x}\left(\int_0^u \sqrt{t^4+x^3}\,dt\right) = \int_0^u \frac{\partial}{\partial x}\sqrt{t^4+x^3}\,dt$$

$$= \int_0^u \frac{1}{2\sqrt{t^4+x^3}}\cdot 3x^2\,dt = \int_0^{x^2} \frac{3x^2}{2\sqrt{t^4+x^3}}\,dt$$

So $\frac{dF}{dx} = 2x\sqrt{x^8+x^3} + \int_0^{x^2} \frac{3x^2}{2\sqrt{t^4+x^3}}\,dt$ //

13.6 Tangent Planes and Normal Lines to Surfaces

In Exercises 1–10, find equations for the

(a) tangent plane and

(b) normal line at the point P_0 on the given surface.

1. $x^2 + y^2 + z^2 = 3, \quad P_0(1,1,1)$
2. $x^2 + y^2 - z^2 = 18, \quad P_0(3,5,-4)$
3. $2z - x^2 = 0, \quad P_0(2,0,2)$
4. $x^2 + 2xy - y^2 + z^2 = 7, \quad P_0(1,-1,3)$
5. $\cos \pi x - x^2y + e^{xz} + yz = 4, \quad P_0(0,1,2)$

Sol'n: $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$.

Then tangent plane at P_0 given by

$$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0.$$

Normal line given by

$$r(t) = (x(t), y(t), z(t)) = \left(x_0 + f_x(P_0)t,\ y_0 + f_y(P_0)t,\ z_0 + f_z(P_0)t\right).$$

$$f(x,y,z) = \cos\pi x - x^2 y + e^{xz} + yz.$$

$$f_x\Big|_{(0,1,2)} = \left(-\pi\sin\pi x - 2xy + ze^{xz}\right)\Big|_{(0,1,2)} = 0 - 2(0)(1) + 2e^{0\cdot 2} = 2$$

$$f_y\Big|_{(0,1,2)} = \left(-x^2 + z\right)\Big|_{(0,1,2)} = 2.$$

$$f_z\Big|_{(0,1,2)} = \left(xe^{xz} + y\right)\Big|_{(0,1,2)} = 1.$$

So tangent plane is given by $2x + 2(y-1) + z - 2 = 0$.

$\Leftrightarrow \; 2x + 2y + z = 4.$

normal line:

$(x, y, z) = (2t,\ 1+2t,\ 2+t)$