MATH 2010

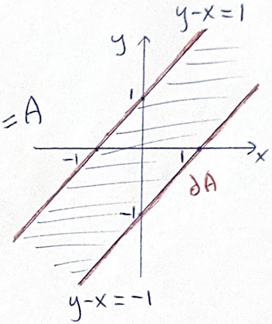
Tutorial 4

Exercises 13.1

Q27 $f(x,y) = \sin^{-1}(y-x)$

(a) find the function's domain.

Domain $= \{(x,y) : -1 \le y-x \le 1\} = A$



(b) find the function's range.

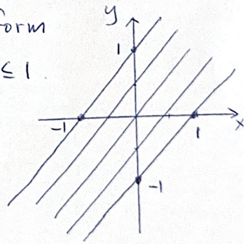
Range $= \{ z : -\frac{\pi}{2} \le z \le \frac{\pi}{2} \}$

(c) describe the function's level curves.

$$f(x,y) = C_0$$
$$\sin^{-1}(y-x) = C_0$$
$$y-x = \sin(C_0)$$
$$y-x = C \quad , \quad C = \sin(C_0) \, , \, -1 \le C \le 1$$

level curves are straight lines of the form

$y - x = C$ where $-1 \le C \le 1$.



(d) find the boundary of the function's domain.

$\partial A = \{(x,y) : y-x = 1$ or $y-x = -1\}$

(e) determine whether the domain is an open region, a closed region, or neither.

S open $\Leftrightarrow S = \mathrm{Int}(S)$

S closed $\Leftrightarrow S = \mathrm{Int}(S) \cup \partial S$

$A = \mathrm{Int}(A) \cup \partial A \ne \mathrm{Int}(A)$

$\therefore$ Domain A is closed, not open.

(f) decide whether the domain is bounded or unbounded.

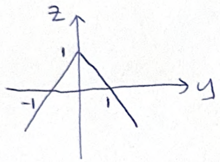
Domain is unbounded.

Q45 $f(x,y) = 1-|y|$

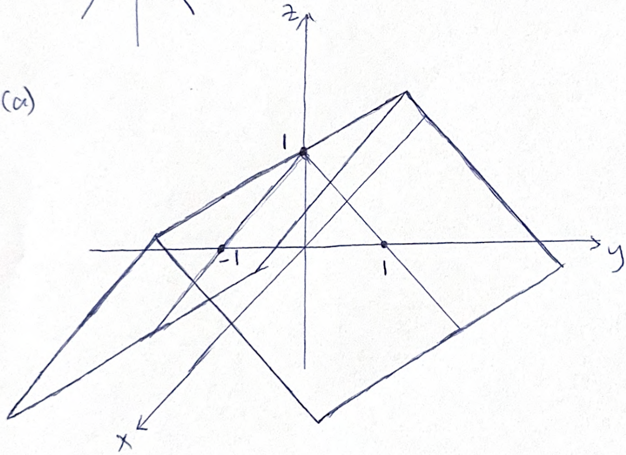
Display the values of the function in two ways:

(a) by sketching the surface $z = f(x,y)$ and

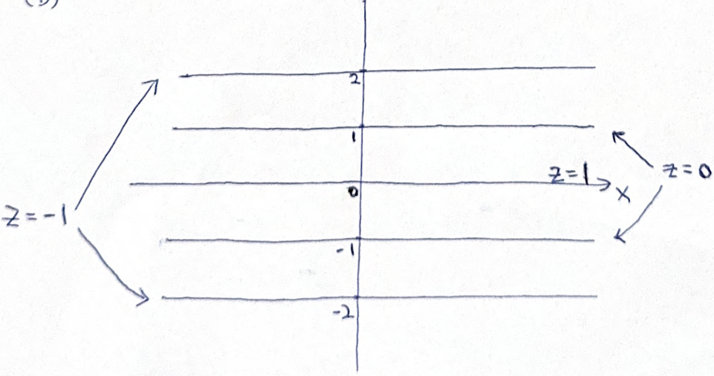
(b) by drawing an assortment of level curves in the function's domain. Label each level curve with its function value.



(a)



(b)



$$\begin{pmatrix} f(x,y)=0 \\ 1-|y|=0 \\ |y|=1 \\ y=\pm 1 \end{pmatrix}$$

Exercises 13.2

Q43 $\quad f(x,y) = \dfrac{x^4 - y^2}{x^4 + y^2}$

By considering different paths of approach, show that the function has no limit as $(x,y) \to (0,0)$.

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L \iff \lim_{\substack{\vec{x}\to\vec{a} \\ \text{along any path}}} f(\vec{x}) = L$$

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) \text{ does not exist} \Leftarrow \lim_{\substack{\vec{x}\to\vec{a} \\ \text{along path 1}}} f(\vec{x}) \neq \lim_{\substack{\vec{x}\to\vec{a} \\ \text{along path 2}}} f(\vec{x})$$

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L \;\cancel{\Leftarrow}\; \lim_{\substack{\vec{x}\to\vec{a} \\ \text{along path 1}}} f(\vec{x}) = \lim_{\substack{\vec{x}\to\vec{a} \\ \text{along path 2}}} f(\vec{x}) = L$$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=kx^2}} \frac{x^4 - y^2}{x^4 + y^2} = \lim_{x\to 0} \frac{x^4 - k^2x^4}{x^4 + k^2x^4} = \lim_{x\to 0} \frac{x^4(1-k^2)}{x^4(1+k^2)}$$

$$= \frac{1-k^2}{1+k^2} \quad \text{different for different values of } k$$

$\Rightarrow$ different limits along paths $y = kx^2$ of different k.

$\Rightarrow \displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - y^2}{x^4 + y^2}$ does not exist.

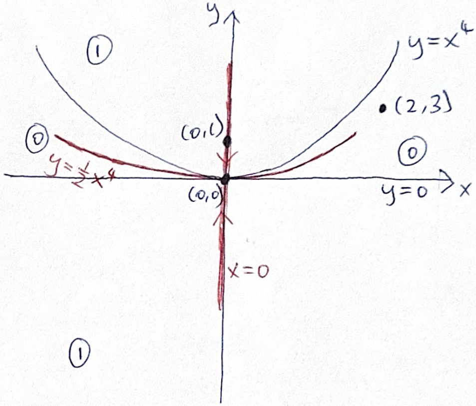
Q55 Let $f(x,y) = \begin{cases} 1, & y \geq x^4 \\ 1, & y \leq 0 \\ 0, & \text{otherwise.} \end{cases}$

Find each of the following limits, or explain that the limit does not exist.

a. $\lim_{(x,y)\to(0,1)} f(x,y)$

b. $\lim_{(x,y)\to(2,3)} f(x,y)$

c. $\lim_{(x,y)\to(0,0)} f(x,y)$



a. When (x,y) near $(0,1)$, it satisfies $y \geq x^4$.

$f(x,y) = 1$ near $(0,1)$.

$$\lim_{(x,y)\to(0,1)} f(x,y) = 1$$

b. $f(x,y) = 0$ near $(2,3)$.

$$\lim_{(x,y)\to(2,3)} f(x,y) = 0$$

c. $\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=0}} f(x,y) = 1$, $\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=\frac{1}{2}x^4}} f(x,y) = 0$

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Q81 $\quad f(x,y,z) = \dfrac{x+y+z}{x^2+y^2+z^2+1}$, $\varepsilon = 0.015$

Show that there exists a $\delta > 0$ such that for all (x,y,z),

$$\underbrace{\sqrt{x^2+y^2+z^2}}_{\|(x,y,z)-(0,0,0)\|} < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| < \varepsilon$$

(about continuity at $(0,0,0)$)

In general : Let $f: A \to \mathbb{R}$, $\vec{a} \in A$.
f is continuous at $\vec{a}$ if
$\forall \varepsilon > 0$, $\exists \delta > 0$ such that
if $\vec{x} \in A$ & $\|\vec{x} - \vec{a}\| < \delta$, then $|f(\vec{x}) - f(\vec{a})| < \varepsilon$.

Take $\delta = 0.005$

$\sqrt{x^2+y^2+z^2} < \delta \Rightarrow |x| < \delta,\ |y| < \delta,\ |z| < \delta.$

$$\Rightarrow |f(x,y,z) - f(0,0,0)| = \left| \frac{x+y+z}{x^2+y^2+z^2+1} - 0 \right|$$

$$= \left| \frac{x+y+z}{x^2+y^2+z^2+1} \right| \leq |x+y+z| \qquad (\because x^2+y^2+z^2+1 \geq 1)$$

$$\leq |x| + |y| + |z| < 3\delta = 3(0.005) = 0.015$$
$$= \varepsilon$$

(For any $\varepsilon > 0$,
take $\delta = \frac{\varepsilon}{3} > 0$.)