MATH2010 Advanced Calculus I

Solution to Homework 10

§13.8

Q2

Solution.
$f(x, y) = xy$
subject to $g(x, y) = x^2 + y^2 - 10 = 0$
Consider $F(x, y, \lambda) = f(x, y) - \lambda g(x, y) = xy - \lambda(x^2 + y^2 - 10)$.

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = y - \lambda(2x) & \Rightarrow y = 2x\lambda \\ 0 = \frac{\partial F}{\partial y} = x - \lambda(2y) & \Rightarrow x = 2y\lambda \\ 0 = \frac{\partial F}{\partial \lambda} = -(x^2 + y^2 - 10) \end{cases}$$

$\Rightarrow x = 4x\lambda^2 \quad \Rightarrow x = 0$ or $\lambda = \pm\frac{1}{2}$
If $x = 0$, then $y = 2x\lambda = 0$. But $x^2 + y^2 - 10 \neq 0$. Hence $x \neq 0$.
$\Rightarrow \lambda = \pm\frac{1}{2}$
$\Rightarrow y = 2x(\pm\frac{1}{2}) = \pm x$
$x^2 + y^2 - 10 = 0 \quad \Rightarrow x^2 + x^2 - 10 = 0 \quad \Rightarrow x = \pm\sqrt{5} \quad \Rightarrow y = \pm\sqrt{5}$
Therefore, f takes on extreme values at $(\sqrt{5}, \sqrt{5}), (\sqrt{5}, -\sqrt{5}), (-\sqrt{5}, \sqrt{5}), (-\sqrt{5}, -\sqrt{5})$.
The extreme values of f are 5 and -5.

□

Q8

Solution.
$f(x, y) = x^2 + y^2$
subject to $g(x, y) = x^2 + xy + y^2 = 1$
Consider $F(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - 1) = x^2 + y^2 - \lambda(x^2 + xy + y^2 - 1)$.

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = 2x - \lambda(2x + y) & \Rightarrow 2x = \lambda(2x + y) \\ 0 = \frac{\partial F}{\partial y} = 2y - \lambda(x + 2y) & \Rightarrow 2y = \lambda(x + 2y) \\ 0 = \frac{\partial F}{\partial \lambda} = -(x^2 + xy + y^2 - 1) \end{cases}$$

$\Rightarrow \lambda = \frac{2y}{x+2y} \quad \Rightarrow 2x = \frac{2y}{x+2y}(2x + y) \quad \Rightarrow x(x + 2y) = y(2x + y) \quad \Rightarrow x^2 = y^2 \quad \Rightarrow y = \pm x$
If $y = x$, $x^2 + x(x) + x^2 - 1 = 0 \quad \Rightarrow 3x^2 = 1 \quad \Rightarrow x = \pm\frac{1}{\sqrt{3}} \quad \Rightarrow y = x = \pm\frac{1}{\sqrt{3}}$
If $y = -x$, $x^2 + x(-x) + x^2 - 1 = 0 \quad \Rightarrow x^2 = 1 \quad \Rightarrow x = \pm 1 \quad \Rightarrow y = -x = \mp 1$
$f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \frac{2}{3}, f(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = \frac{2}{3}, f(1, -1) = 2, f(-1, 1) = 2$
Therefore, $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ are nearest to the origin; $(1, -1)$ and $(-1, 1)$ are farthest from the origin. □

Q12

Solution.
Let (x, y) be the vertex of the rectangle in the quadrant I.
$f(x, y) = 4x + 4y$
subject to $g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Consider $F(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - 1) = 4x + 4y - \lambda(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1)$.

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = 4 - \lambda(\frac{2x}{a^2}) & \Rightarrow 4 = \lambda(\frac{2x}{a^2}) \\ 0 = \frac{\partial F}{\partial y} = 4 - \lambda(\frac{2y}{b^2}) & \Rightarrow 4 = \lambda(\frac{2y}{b^2}) \\ 0 = \frac{\partial F}{\partial \lambda} = -(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1) \end{cases}$$

$\Rightarrow \lambda = \frac{2a^2}{x} \quad \Rightarrow 4 = (\frac{2a^2}{x})(\frac{2y}{b^2}) \quad \Rightarrow y = (\frac{b^2}{a^2})x$

$\Rightarrow \frac{x^2}{a^2} + \frac{((\frac{b^2}{a^2})x)^2}{b^2} - 1 = 0 \quad \Rightarrow \frac{x^2}{a^2} + \frac{b^2x^2}{a^4} = 1 \quad \Rightarrow (a^2+b^2)x^2 = a^4 \quad \Rightarrow x = \frac{a^2}{\sqrt{a^2+b^2}}$ since $x > 0 \quad \Rightarrow y = (\frac{b^2}{a^2})x = \frac{b^2}{\sqrt{a^2+b^2}}$

Therefore, dimensions of the rectangle of largest perimeter:

width $= 2x = \frac{2a^2}{\sqrt{a^2+b^2}}$, height $= 2y = \frac{2b^2}{\sqrt{a^2+b^2}}$.

The largest perimeter $= 4x + 4y = \frac{4a^2+4b^2}{\sqrt{a^2+b^2}} = 4\sqrt{a^2+b^2}$ □

Q18

Solution.

$f(x,y,z) = (x-1)^2 + (y+1)^2 + (z-1)^2$

subject to $g(x,y,z) = x^2 + y^2 + z^2 = 4$

Consider $F(x,y,z,\lambda) = f(x,y,z) - \lambda(g(x,y,z) - 4) = (x-1)^2 + (y+1)^2 + (z-1)^2 - \lambda(x^2+y^2+z^2-4)$.

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = 2(x-1) - \lambda(2x) & \Rightarrow x - 1 = \lambda x & \Rightarrow x = \frac{1}{1-\lambda} \\ 0 = \frac{\partial F}{\partial y} = 2(y+1) - \lambda(2y) & \Rightarrow y + 1 = \lambda y & \Rightarrow y = -\frac{1}{1-\lambda} \\ 0 = \frac{\partial F}{\partial z} = 2(z-1) - \lambda(2z) & \Rightarrow z - 1 = \lambda z & \Rightarrow z = \frac{1}{1-\lambda} \\ 0 = \frac{\partial F}{\partial \lambda} = -(x^2+y^2+z^2-4) \end{cases}$$

$\Rightarrow (\frac{1}{1-\lambda})^2 + (\frac{1}{1-\lambda})^2 + (\frac{1}{1-\lambda})^2 = 4 \quad \Rightarrow \frac{1}{1-\lambda} = \pm\frac{2}{\sqrt{3}}$

$\Rightarrow (x,y,z) = (\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$ or $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$

The largest value of f occurs at the point $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$.

Therefore, $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$ is farthest from the point $(1,-1,1)$. □

Q24

Solution.

$f(x,y,z) = x + 2y + 3z$

subject to $g(x,y,z) = x^2 + y^2 + z^2 = 25$

Consider $F(x,y,z,\lambda) = f(x,y,z) - \lambda(g(x,y,z) - 25) = x + 2y + 3z - \lambda(x^2+y^2+z^2-25)$.

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = 1 - \lambda(2x) & \Rightarrow 1 = 2\lambda x & \Rightarrow x = \frac{1}{2\lambda} \\ 0 = \frac{\partial F}{\partial y} = 2 - \lambda(2y) & \Rightarrow 2 = 2\lambda y & \Rightarrow y = \frac{1}{\lambda} \\ 0 = \frac{\partial F}{\partial z} = 3 - \lambda(2z) & \Rightarrow 3 = 2\lambda z & \Rightarrow z = \frac{3}{2\lambda} \\ 0 = \frac{\partial F}{\partial \lambda} = -(x^2+y^2+z^2-25) \end{cases}$$

$\Rightarrow y = 2x, z = 3x$

$\Rightarrow x^2 + (2x)^2 + (3x)^2 - 25 = 0 \quad \Rightarrow 14x^2 = 25 \quad \Rightarrow x = \pm\frac{5}{\sqrt{14}}$

$\Rightarrow (x,y,z) = (\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}})$ or $(-\frac{5}{\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{15}{\sqrt{14}})$

$f(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}) = \frac{70}{\sqrt{14}} = 5\sqrt{14}, f(-\frac{5}{\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{15}{\sqrt{14}}) = -\frac{70}{\sqrt{14}} = -5\sqrt{14}$

Therefore, f has its maximum value $5\sqrt{14}$ at $(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}})$ and its minimum value $-5\sqrt{14}$ at $(-\frac{5}{\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{15}{\sqrt{14}})$. □

Q38

Solution.

$f(x,y,z) = x^2 + y^2 + z^2$

subject to $g_1(x,y,z) = x + 2y + 3z = 6$ and $g_2(x,y,z) = x + 3y + 9z = 9$

Consider $F(x,y,z,\lambda_1,\lambda_2) = f(x,y,z) - \lambda_1(g_1(x,y,z) - 6) - \lambda_2(g_2(x,y,z) - 9)$

$= x^2 + y^2 + z^2 - \lambda_1(x + 2y + 3z - 6) - \lambda_2(x + 3y + 9z - 9)$.

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = 2x - \lambda_1(1) - \lambda_2(1) & \Rightarrow 2x = \lambda_1 + \lambda_2 \\ 0 = \frac{\partial F}{\partial y} = 2y - \lambda_1(2) - \lambda_2(3) & \Rightarrow 2y = 2\lambda_1 + 3\lambda_2 \\ 0 = \frac{\partial F}{\partial z} = 2z - \lambda_1(3) - \lambda_2(9) & \Rightarrow 2z = 3\lambda_1 + 9\lambda_2 \\ 0 = \frac{\partial F}{\partial \lambda_1} = -(x + 2y + 3z - 6) \\ 0 = \frac{\partial F}{\partial \lambda_2} = -(x + 3y + 9z - 9) \end{cases}$$

$x + 2y + 3z - 6 = 0 \quad \Rightarrow \frac{1}{2}(\lambda_1 + \lambda_2) + (2\lambda_1 + 3\lambda_2) + \frac{3}{2}(3\lambda_1 + 9\lambda_2) = 6 \quad \Rightarrow 7\lambda_1 + 17\lambda_2 = 6$
$x + 3y + 9z - 9 = 0 \quad \Rightarrow \frac{1}{2}(\lambda_1 + \lambda_2) + \frac{3}{2}(2\lambda_1 + 3\lambda_2) + \frac{9}{2}(3\lambda_1 + 9\lambda_2) = 9 \quad \Rightarrow 17\lambda_1 + \frac{91}{2}\lambda_2 = 9$
$\Rightarrow \lambda_1 = \frac{240}{59}, \lambda_2 = -\frac{78}{59}$
$\Rightarrow x = \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{81}{59}, y = \frac{1}{2}(2\lambda_1 + 3\lambda_2) = \frac{123}{59}, z = \frac{1}{2}(3\lambda_1 + 9\lambda_2) = \frac{9}{59}$
Therefore, the minimum value is $f(\frac{81}{59}, \frac{123}{59}, \frac{9}{59}) = \frac{21771}{59^2} = \frac{369}{59}$. □

Q40

Solution.
$f(x, y, z) = 2x^2 + yz$
subject to $g_1(x, y, z) = x^2 + z^2 = 9$ and $g_2(x, y, z) = y - z = 4$
Consider $F(x, y, z, \lambda_1, \lambda_2) = f(x, y, z) - \lambda_1(g_1(x, y, z) - 9) - \lambda_2(g_2(x, y, z) - 4) = 2x^2 + yz - \lambda_1(x^2 + z^2 - 9) - \lambda_2(y - z - 4)$.

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = 4x - \lambda_1(2x) - \lambda_2(0) & \Rightarrow 2x = \lambda_1 x \quad \Rightarrow x = 0 \text{ or } \lambda_1 = 2 \\ 0 = \frac{\partial F}{\partial y} = z - \lambda_1(0) - \lambda_2(1) & \Rightarrow z = \lambda_2 \\ 0 = \frac{\partial F}{\partial z} = y - \lambda_1(2z) - \lambda_2(-1) & \Rightarrow y = 2\lambda_1 z - \lambda_2 \\ 0 = \frac{\partial F}{\partial \lambda_1} = -(x^2 + z^2 - 9) \\ 0 = \frac{\partial F}{\partial \lambda_2} = -(y - z - 4) \end{cases}$$

If $x = 0$, $x^2 + z^2 - 9 = 0 \quad \Rightarrow z = \pm 3$
$y - z - 4 = 0 \quad \Rightarrow y = 4 + z \quad \Rightarrow (x, y, z) = (0, 7, 3)$ or $(0, 1, -3)$
If $x \neq 0$, then $\lambda_1 = 2 \quad \Rightarrow y = 4z - \lambda_2 = 4\lambda_2 - \lambda_2 = 3\lambda_2$
$y - z - 4 = 0 \quad \Rightarrow 3\lambda_2 - \lambda_2 = 4 \quad \Rightarrow \lambda_2 = 2 \quad \Rightarrow y = 6, z = 2$
$x^2 + z^2 - 9 = 0 \quad \Rightarrow x^2 + 4 - 9 = 0 \quad \Rightarrow x = \pm\sqrt{5} \quad \Rightarrow (x, y, z) = (\sqrt{5}, 6, 2)$ or $(-\sqrt{5}, 6, 2)$
$f((0, 7, 3) = 21, f(0, 1, -3) = -3, f(\sqrt{5}, 6, 2) = 22, f(-\sqrt{5}, 6, 2) = 22$
Maximum value = 22, Minimum value = −3. □

§13.9

Q2

Solution.
$f(x, y) = e^x \cos y$
$\nabla f(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \end{pmatrix}$
$Hf(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ -e^x \sin y & -e^x \cos y \end{pmatrix}$
$f(0, 0) = 1, \nabla f(0, 0) = \begin{pmatrix} 1 & 0 \end{pmatrix}, Hf(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$P_2(x, y) = 1 + \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x - 0 \\ y - 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x - 0 & y - 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x - 0 \\ y - 0 \end{pmatrix} = 1 + x + \frac{1}{2}\begin{pmatrix} x & -y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2$ □

Q4

Solution.
$f(x, y) = \sin x \cos y$
$\nabla f(x, y) = \begin{pmatrix} \cos x \cos y & -\sin x \sin y \end{pmatrix}$
$Hf(x, y) = \begin{pmatrix} -\sin x \cos y & -\cos x \sin y \\ -\cos x \sin y & -\sin x \cos y \end{pmatrix}$

$f(0,0) = 0, \nabla f(0,0) = \begin{pmatrix} 1 & 0 \end{pmatrix}, Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$P_2(x,y) = 0 + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x-0 \\ y-0 \end{pmatrix} + 0 = x$ □

Q10

Solution.

$f(x,y) = \frac{1}{1-x-y+xy}$

$\nabla f(x,y) = \begin{pmatrix} -\frac{-1+y}{(1-x-y+xy)^2} & -\frac{-1+x}{(1-x-y+xy)^2} \end{pmatrix} = \begin{pmatrix} \frac{1}{(1-x)^2(1-y)} & \frac{1}{(1-x)(1-y)^2} \end{pmatrix}$

$Hf(x,y) = \begin{pmatrix} \frac{2}{(1-x)^3(1-y)} & \frac{1}{(1-x)^2(1-y)^2} \\ \frac{1}{(1-x)^2(1-y)^2} & \frac{2}{(1-x)(1-y)^3} \end{pmatrix}$

$f(-1,0) = \frac{1}{2}, \nabla f(-1,0) = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \end{pmatrix}, Hf(-1,0) = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 1 \end{pmatrix}$

$P_2(x,y) = \frac{1}{2} + \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x+1 \\ y-0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x+1 & y-0 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} x+1 \\ y-0 \end{pmatrix} = \frac{1}{2} + \frac{1}{4}x + \frac{1}{4} + \frac{1}{2}y + \frac{1}{2} \begin{pmatrix} \frac{1}{4}x + \frac{1}{4} + \frac{1}{4}y & \frac{1}{4}x + \frac{1}{4} + y \end{pmatrix} \begin{pmatrix} x+1 \\ y \end{pmatrix}$

$= \frac{1}{2} + \frac{1}{4}x + \frac{1}{4} + \frac{1}{2}y + \frac{1}{2}(\frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{4}xy + \frac{1}{4}x + \frac{1}{4} + \frac{1}{4}y + \frac{1}{4}xy + \frac{1}{4}y + y^2)$

$= \frac{7}{8} + \frac{1}{2}x + \frac{3}{4}y + \frac{1}{4}xy + \frac{1}{8}x^2 + \frac{1}{2}y^2$ □