Thm (Implicit Function Theorem)

Let $\Omega \subseteq \mathbb{R}^{n+k}$ be open, $\vec{F}: \Omega \to \mathbb{R}^k$, $\vec{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_k \end{bmatrix}$ be $C^1$

Denote $\vec{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ & $\vec{y} = (y_1, \cdots, y_k) \in \mathbb{R}^k$.

$$\vec{F}(\vec{x}, \vec{y}) = \begin{bmatrix} F_1(\vec{x}, \vec{y}) \\ \vdots \\ F_k(\vec{x}, \vec{y}) \end{bmatrix} = \begin{bmatrix} F_1(x_1, \cdots, x_n, y_1, \cdots, y_k) \\ \vdots \\ F_k(x_1, \cdots, x_n, y_1, \cdots, y_k) \end{bmatrix}$$

Suppose $(\vec{a}, \vec{b}) \in \Omega$, where $\vec{a} \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^k$ such that

$$\vec{F}(\vec{a}, \vec{b}) = \vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$$

and the $k \times k$ matrix

$$\left[ \frac{\partial F_i}{\partial y_j}(\vec{a}, \vec{b}) \right]_{1 \le i, j \le k} = \begin{bmatrix} \frac{\partial F_1}{\partial y_1}(\vec{a}, \vec{b}) & \cdots & \frac{\partial F_1}{\partial y_k}(\vec{a}, \vec{b}) \\ \vdots & & \vdots \\ \frac{\partial F_k}{\partial y_1}(\vec{a}, \vec{b}) & \cdots & \frac{\partial F_k}{\partial y_k}(\vec{a}, \vec{b}) \end{bmatrix}$$

is invertible (i.e. $\det \left[ \frac{\partial F_i}{\partial y_j}(\vec{a}, \vec{b}) \right] \neq 0$)

Then there are open sets $U \subseteq \mathbb{R}^n$ containing $\vec{a}$, and $V \subseteq \mathbb{R}^k$ containing $\vec{b}$ such that there exists a unique function $\vec{\varphi}: U \to V$ with $\vec{\varphi}(\vec{a}) = \vec{b}$ and

$$\vec{F}(\vec{x}, \vec{\varphi}(\vec{x})) = \vec{c}, \quad \forall \vec{x} \in U$$

Moreover, $\vec{\varphi}$ is $C^1$ and (by implicit differentiation)

$$\left[ \frac{\partial \varphi_j}{\partial x_\ell}(\vec{x}) \right]_{k \times n} = - \left[ \frac{\partial F_i}{\partial y_j}(\vec{x}, \vec{\varphi}(\vec{x})) \right]_{k \times k}^{-1} \left[ \frac{\partial F_i}{\partial x_\ell}(\vec{x}, \vec{\varphi}(\vec{x})) \right]_{k \times n}$$

(Pf: in MATH3060)

eg: If $\vec{F}$ is actually linear:

$$\begin{cases} F_1(x_1,\cdots,x_n,y_1,\cdots,y_k) = c_{11}x_1+\cdots+c_{1n}x_n + d_{11}y_1+\cdots+d_{1k}y_k \\ \vdots \\ F_k(x_1,\cdots,x_n,y_1,\cdots,y_k) = c_{k1}x_1+\cdots+c_{kn}x_n + d_{k1}y_1+\cdots+d_{kk}y_k \end{cases}$$

Then $\vec{F}(\vec{x},\vec{y}) = \vec{c}$ can be written

$$\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{k1} & \cdots & c_{kn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} d_{11} & \cdots & d_{1k} \\ \vdots & & \vdots \\ d_{k1} & \cdots & d_{kk} \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$$

And

$$\left(\frac{\partial F_i}{\partial y_j}\right)_{1\le i,j\le n} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_k} \\ \vdots & & \vdots \\ \frac{\partial F_k}{\partial y_1} & \cdots & \frac{\partial F_k}{\partial y_k} \end{pmatrix} = \begin{pmatrix} d_{11} & \cdots & d_{1k} \\ \vdots & & \vdots \\ d_{k1} & \cdots & d_{kk} \end{pmatrix}$$

If $\left(\frac{\partial F_i}{\partial y_j}\right) = (d_{ij})$ is invertible (i.e. $\det(d_{ij}) \neq 0$),

then
$$\begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} = \begin{pmatrix} d_{11} & \cdots & d_{1k} \\ \vdots & & \vdots \\ d_{k1} & \cdots & d_{kk} \end{pmatrix}^{-1}\left[-\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{k1} & \cdots & c_{kn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}\right]$$

$= \vec{\varphi}(\vec{x})$ is the required implicit function

(same $\vec{\varphi}$ for all $(\vec{a},\vec{b})$ with $\vec{F}(\vec{a},\vec{b}) = \vec{c}$, & here $U = \mathbb{R}^n$, $V = \mathbb{R}^k$)

Clearly
$$\left(\frac{\partial \varphi_j}{\partial x_l}\right)_{k\times n} = \left(\frac{\partial y_j}{\partial x_l}\right)_{k\times n} = -\begin{pmatrix} d_{11} & \cdots & d_{1k} \\ \vdots & & \vdots \\ d_{k1} & \cdots & d_{kk} \end{pmatrix}^{-1}\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{k1} & \cdots & c_{kn} \end{pmatrix}$$

$\left(\frac{\partial F_i}{\partial x_l}\right)_{k\times n}$

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Special case (A) : $k=1$ (1 constraint)

$$F : \Omega \subset \mathbb{R}^{n+1} \to \mathbb{R}$$

$$F(\vec{x}, \vec{y}) = F(x_1, \dots, x_n, y) = c \qquad \text{(1 constraint)}$$

Suppose $\vec{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$, $b \in \mathbb{R}$ s.t.

$$F(a_1, \cdots, a_n, b) = c$$

IFT : If $\frac{\partial F}{\partial y}(a_1, \cdots, a_n, b) \neq 0$,

then $\exists$ $U^{\text{open in } \mathbb{R}^n}$ s.t. $(a_1, \cdots, a_n) \in U$ and

$V^{\text{open in } \mathbb{R}}$ s.t. $b \in V$

and $\exists$ unique $C^1$ $\varphi : U \to V$ s.t.

$$\varphi(a_1, \cdots, a_n) = b \quad \&$$

$$F(x_1, \cdots, x_n, \varphi(x_1, \cdots, x_n)) = c, \ \forall (x_1, \cdots, x_n) \in U$$

(i.e. $y = \varphi(x_1, \cdots, x_n)$ solves the constraint $F(x_1, \cdots, x_n, y) = c$ "near" $(a_1, \cdots, a_n, b)$)

Moreover, $\frac{\partial \varphi}{\partial x_i}$ can be calculated using implicit differentiation.

In eg2: $x^2+y^2+z^2=2$ Solve $z=z(x,y)$

| $(x,y,z)$<br>$\mathbb{R}^3$ notation | $(x_1,x_2,y)$<br>general notation (this "y" is not the "y" on the other side) |
|---|---|
| $g(x,y,z)=x^2+y^2+z^2=2$<br>near $(0,1,1)$ | $F(x_1,x_2,y)=x_1^2+x_2^2+y^2=c \quad (c=2)$<br>$\vec{a}=(a_1,a_2)=(0,1)$, $b=1$ |
| $\frac{\partial g}{\partial z}(0,1,1)=2\neq 0$<br>By IFT<br>$\exists\ z=z(x,y)$ "near" $(0,1)$<br>s.t. $\begin{cases} g(x,y,z(x,y))=2 \\ z(0,1)=1 \end{cases}$<br>$x^2+y^2+(z(x,y))^2=2$ | $\frac{\partial F}{\partial y}(a_1,a_2,b)=2\neq 0$<br>By IFT<br>$\exists\ y=\varphi(x_1,x_2)$ "near" $(a_1,a_2,b)$<br>s.t. $\begin{cases} F(x_1,x_2,\varphi(x_1,x_2))=c \quad (c=2) \\ \varphi(a_1,a_2)=b \end{cases}$<br>$\uparrow \varphi(0,1)=1$<br>$x_1^2+x_2^2+(\varphi(x_1,x_2))^2=c$ |
| $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ can be computed<br>by implicit differentiation | $\frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}$ can be computed<br>by implicit differentiation |

Special case (B) $\quad n=1,\ k=2 \quad$ (2-constraints)

$$\vec{F} : \Omega \subset \mathbb{R}^{1+2} \longrightarrow \mathbb{R}^2$$

$$\vec{F}(x, y_1, y_2) = \begin{bmatrix} F_1(x, y_1, y_2) \\ F_2(x, y_1, y_2) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{c}$$

Suppose $(a, b_1, b_2)$ satisfies the constraints $\vec{F}(a, b_1, b_2) = \vec{c}$,

then IFT means:

if $\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix}(a, b_1, b_2)$ is <u>invertible</u> (ie $\det \neq 0$)

then $\exists$ $y_1 = \varphi_1(x)$ & $y_2 = \varphi_2(x)$ "near" $(a, b_1, b_2)$

solving the constraints (locally)

$$\begin{cases} F_1(x, \varphi_1(x), \varphi_2(x)) = c_1 \\ F_2(x, \varphi_1(x), \varphi_2(x)) = c_2 \end{cases}$$

& $\begin{cases} \varphi_1(a) = b_1 \\ \varphi_2(a) = b_2 \end{cases}$

eg3 $\begin{cases} x^2+y^2+z^2=2 \\ x+z=1 \end{cases}$ Solve for $y=y(x)$, $z=z(x)$? near $(0,1,1)$

$(x,y,z)$ $\longleftrightarrow$ $(x,y_1,y_2)$

$\mathbb{R}^3$ notation | General Notation

$\begin{cases} g(x,y,z)=x^2+y^2+z^2=2 \\ h(x,y,z) = x+z = 1 \end{cases}$

near $(0,1,1)$

$$\begin{pmatrix} \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix}_{(0,1,1)} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

$\underline{\text{invertible}}$
$\det = 2 \neq 0$

By IFT

$\exists\ y=y(x)$, $z=z(x)$ "near" $(0,1,1)$ s.t

$\begin{cases} g(x,y(x),z(x))=2 \\ h(x,y(x),z(x))=1 \end{cases}$

$\begin{cases} y(0)=1 \\ z(0)=1 \end{cases}$ $\begin{pmatrix} y(x)=\sqrt{2-x^2-(1-x)^2} \\ z(x)=1-x \end{pmatrix}$

$\left( x^2+y^2(x)+z^2(x)=2 \ \& \ x+z(x)=1 \right)$

$\begin{cases} F_1(x,y_1,y_2)=x^2+y_1^2+y_2^2 = c_1 & (c_1=2) \\ F_2(x,y_1,y_2) = x+y_2 = c_2 & (c_2=1) \end{cases}$

$a=0$, $\vec{b}=(b_1,b_2)=(1,1)$, $\vec{c}=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{pmatrix}_{(a,b_1,b_2)} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

$\underline{\text{invertible}}$
$\det = 2 \neq 0$

By IFT,

$\exists\ y_1=\varphi_1(x)$, $y_2=\varphi_2(x)$ "near" $(a,b_1,b_2)$ s.t.

$\begin{cases} F_1(x,\varphi_1(x),\varphi_2(x))=c_1 \\ F_2(x,\varphi_1(x),\varphi_2(x))=c_2 \end{cases}$

$\begin{cases} \varphi_1(a)=b_1 \\ \varphi_2(a)=b_2 \end{cases}$

Remark: $\left.\frac{dy}{dx}, \frac{dz}{dx}\right|_{x=0}$ can be calculated by implicit differentiation.

Remark: $\left.\frac{d\varphi_1}{dx}, \frac{d\varphi_2}{dx}\right|_{x=a}$ can be calculated by implicit differentiation.

eg: Consider the constraints

$$\begin{cases} xz + \sin(yz - x^2) = 8 \\ x + 4y + 3z = 18 \end{cases}$$

$(2,1,4)$ is a solution.

Can we solve 2 of the variables as functions of the remaining variable?

<u>Soln</u>:

$$\vec{F}(x,y,z) = \begin{bmatrix} F_1(x,y,z) \\ F_2(x,y,z) \end{bmatrix} = \begin{bmatrix} xz + \sin(yz - x^2) \\ x + 4y + 3z \end{bmatrix}$$

$$D\vec{F} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} z - 2x\cos(yz - x^2) & z\cos(yz - x^2) & x + y\cos(yz - x^2) \\ 1 & 4 & 3 \end{bmatrix}$$

$$\Rightarrow D\vec{F}(2,1,4) = \begin{bmatrix} 0 & 4 & 3 \\ 1 & 4 & 3 \end{bmatrix}$$ (check!)

$$\begin{array}{cc} x & y \\ \downarrow & \downarrow \end{array} \qquad \begin{array}{cc} x & z \\ \downarrow & \downarrow \end{array} \qquad \begin{array}{cc} y & z \\ \downarrow & \downarrow \end{array}$$

$$\det\begin{pmatrix} 0 & 4 \\ 1 & 4 \end{pmatrix} \qquad \det\begin{pmatrix} 0 & 3 \\ 1 & 3 \end{pmatrix} \qquad \det\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix}$$

$$\begin{array}{ccc} \| & \| & \| \\ -4 \neq 0 & -3 \neq 0 & 0 \end{array}$$

IFT $\Rightarrow$

- $x, y$ can be solved as (diff.) functions
  $$x = x(z) \ \& \ y = y(z)$$
  of $z$ near $(2,1,4)$

- $x, z$ can be solved as (diff.) functions
  $$x = x(y) \ \& \ z = z(y)$$
  of $y$ near $(2,1,4)$

- <u>No conclusion</u> on whether $y, z$ can be solved as (diff.) functions of $x$ near $(2,1,4)$.

(Further analysis for this particular example:

implicit diff. $\Rightarrow$ if $y(x)$ & $z(x)$ exists & diff.

then $\begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ which is a contradiction. $\Rightarrow$ $y, z$ cannot be solved as (diff.) functions of $x$)

**Thm** (Inverse Function Theorem)

Let $\vec{f}: \Omega \to \mathbb{R}^n$ be $C^1$, ($\Omega \subset \mathbb{R}^n$, open)

Suppose $D\vec{f}(\vec{a})$ is invertible ($n \times n$ matrix)

Then $\exists$ open sets $U \subseteq \mathbb{R}^n$ containing $\vec{a}$,

$V \subseteq \mathbb{R}^n$ containing $\vec{b} = \vec{f}(\vec{a})$

such that $\exists$ a unique function

$$\vec{g}: V \to U \quad \text{with}$$

$$\vec{g}(\vec{b}) = \vec{a}$$

satisfying $\begin{cases} \vec{g}(\vec{f}(\vec{x})) = \vec{x}, & \forall \vec{x} \in U \\ \vec{f}(\vec{g}(\vec{y})) = \vec{y}, & \forall \vec{y} \in V \end{cases}$ (i.e. $\vec{g} = (\vec{f}|_U)^{-1}$)

Moreover, $\vec{g}$ is $C^1$ and

$$D\vec{g}(\vec{y}) = \left[D\vec{f}(\vec{g}(\vec{y}))\right]^{-1}, \quad \forall \vec{y} \in V.$$

(Pf: MATH3060)

Remark: $\vec{g} = (\vec{f}|_U)^{-1}$ is called a local inverse of $\vec{f}$ at $\vec{a}$.

eg: $\vec{f}$ is actually linear:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

Clearly $\vec{f}$ is invertible $\Leftrightarrow$ $\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}$ invertible

$$\vec{g}(\vec{y}) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}^{-1} \left[ \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} - \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \right] = f^{-1}$$

But $\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} = D\vec{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$ $\therefore$ the condition required by IFT is satisfied.

( Same as the linear example of Implicit Function Thm, $U = \mathbb{R}^n = V$ )

$$D\vec{g} = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial g_n}{\partial y_1} & \cdots & \frac{\partial g_n}{\partial y_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

$$= \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}^{-1} = (D\vec{f})^{-1}$$

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