Implicit Function Theorem

Recall: Implicit differentiation

e.g. $x^2+y^2+z^2=2$ and if $z=z(x,y)$,

then
$$\begin{cases} \dfrac{\partial}{\partial x}(x^2+y^2+z^2)=0 \Rightarrow 2x+2z\dfrac{\partial z}{\partial x}=0 \\[2ex] \dfrac{\partial}{\partial y}(x^2+y^2+z^2)=0 \Rightarrow 2y+2z\dfrac{\partial z}{\partial y}=0 \end{cases}$$

If the point (x,y,z) satisfies $z\neq 0$,

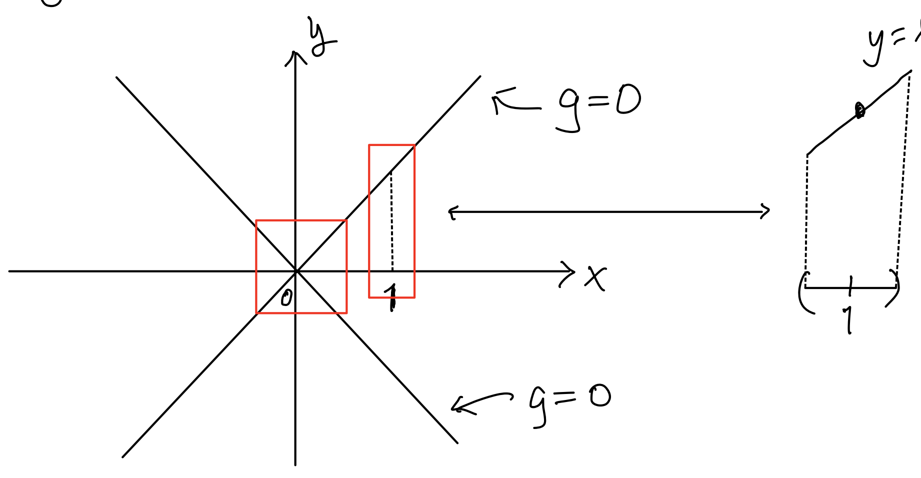
then we have $\dfrac{\partial z}{\partial x}=-\dfrac{x}{z}$ & $\dfrac{\partial z}{\partial y}=-\dfrac{y}{z}$.

Question: If a level set $g(x,y)=c$ (or more generally) is given, can we "solve" the constraint?

i.e. can we find $y=h(x)$ s.t. $g(x,h(x))=c$

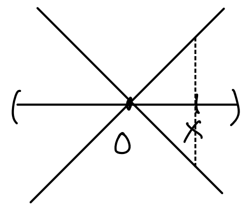
or $x=k(y)$ s.t. $g(k(y),y)=c$?

eg 1 $g(x,y)=x^2-y^2=0$ $(\Rightarrow x=\pm y)$



Yes, we can solve for $y=h(x)$ near $(x,y)=(1,1)$

But near $(x,y)=(0,0)$



2 y-values $\longleftrightarrow$ 1 x-value

It cannot be a graph of a function $y=h(x)$

$\Rightarrow$ Cannot solve for a function $y=h(x)$ near $(0,0)$.

<u>eg 2</u> $S: x^2+y^2+z^2=2$ in $\mathbb{R}^3$

Can we solve $z=h(x,y)$ near $(0,1,1)$?

Can we solve $x=k(y,z)$ near $(0,1,1)$?

<u>Observations:</u>

1^{st} eqt. : if $z=h(x,y)$ exists, then

$$\begin{cases} \partial_x(x^2+y^2+z^2)=0 & \Rightarrow \quad \dfrac{\partial z}{\partial x}=-\dfrac{x}{z} \\ \partial_y(x^2+y^2+z^2)=0 & \Rightarrow \quad \dfrac{\partial z}{\partial y}=-\dfrac{y}{z} \end{cases} \quad \text{provided } z\neq 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x}(0,1,1)=0, \quad \frac{\partial z}{\partial y}(0,1,1)=-1$$

At least, there is no contradiction & we have hope to solve it!

2^{nd} eqt: if $x=k(x,y)$ exists,

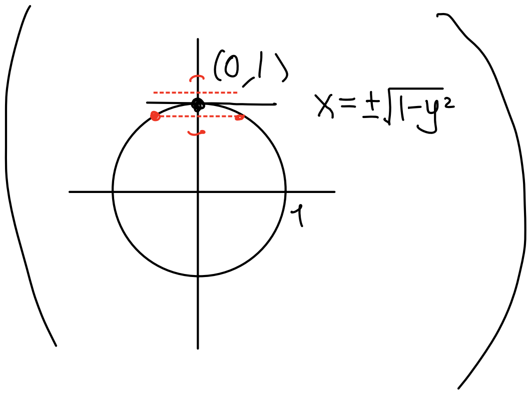
$$\begin{cases} \partial_y(x^2+y^2+z^2)=0 \\ \partial_z(x^2+y^2+z^2)=0 \end{cases} \Rightarrow \begin{cases} 2x\dfrac{\partial x}{\partial y}+2y=0 \\ 2x\dfrac{\partial x}{\partial z}+2z=0 \end{cases}$$

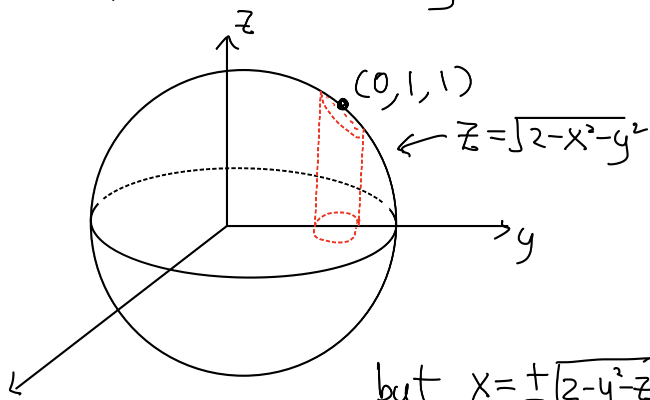
At the point $(0,1,1)$, we have $\begin{cases} 0+2=0 \\ 0+2=0 \end{cases}$

which is a contradiction.

So there exists <u>NO</u> $x = k(y,z)$ (which is differentiable)

near the point $(x,y,z) = (0,1,1)$ solving the constraint.





but $x = \pm\sqrt{2-y^2-z^2}$

two values in near (0,1,1)

General situation (in 3-variables)

$$F(x,y,z) = c$$

If $z = z(x,y)$ (differentiable), then implicit differentiation

$$\begin{array}{ll} \frac{\partial}{\partial x}: & \left\{ \begin{array}{l} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \\ \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \end{array} \right. \\ \frac{\partial}{\partial y}: & \end{array}$$

If $F(\vec{a}) = c$ & $\underline{\underline{\frac{\partial F}{\partial z}(\vec{a}) \neq 0}}$, then

$$\begin{bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{bmatrix} = -\frac{1}{\frac{\partial F}{\partial z}(\vec{a})} \begin{bmatrix} \frac{\partial F}{\partial x}(\vec{a}) \\ \frac{\partial F}{\partial y}(\vec{a}) \end{bmatrix}$$

(at (x_0, y_0) if $\vec{a} = (x_0, y_0, z_0)$)

eg3 (Multiple constraints)

$$\mathcal{C} \left\{ \begin{array}{l} x^2 + y^2 + z^2 = 2 \\ x + z = 1 \end{array} \right.$$

(3-variables, 2 equations expect $\mathcal{C}$ is "1-dim")

Question: Can we solve $\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y(x) \\ z(x) \end{bmatrix}$?

Observation: If we have $y=y(x)$ & $z=z(x)$, (both differentiable)

differentiation wrt $x \Rightarrow$

$$\frac{d}{dx}\left(x^2+(y(x))^2+(z(x))^2\right)=0$$

$$\Rightarrow \quad 2x+2y\frac{dy}{dx}+2z\frac{dz}{dx}=0$$

$$y\frac{dy}{dx}+z\frac{dz}{dx}=-x \qquad \text{———} (1)$$

&

$$\frac{d}{dx}(x+z(x))=0$$

$$1+\frac{dz}{dx}=0 \qquad \text{———} (2)$$

$$\therefore \begin{bmatrix} y & z \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix}=\begin{bmatrix} -x \\ -1 \end{bmatrix}$$

If $\underline{\underline{\det\begin{bmatrix} y & z \\ 0 & 1 \end{bmatrix}\neq 0}}$, then one can solve (uniquely)

for $\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix}=\begin{bmatrix} y & z \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -x \\ -1 \end{bmatrix}$

So we have hope to the existence of $\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} y(x) \\ z(x) \end{bmatrix}$.

For instance $(x,y,z)=(0,1,1)$ (on $\mathcal{C}$)

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix}=\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is solvable since $\det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}=1\neq 0$.

and $\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -1 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (check!)

$\#$

In general, given $\mathcal{C}: \begin{cases} F_1(x,y,z) = C_1 \\ F_2(x,y,z) = C_2 \end{cases}$

$$\left(\vec{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix},\ \vec{F}(\vec{x}) = \vec{C},\ \vec{F}: \mathbb{R}^3 \to \mathbb{R}^2 \right)$$

Suppose $F_i(a,b,c) = C_i$, $i=1,2$

Assume $y = y(x)$, $z = z(x)$ near (a,b,c) (diff.)

(Implicit differentiation)
$$\begin{cases} \frac{d}{dx} F_1(x,y,z) = 0 \\ \frac{d}{dx} F_2(x,y,z) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}\frac{dy}{dx} + \frac{\partial F_1}{\partial z}\frac{dz}{dx} = 0 \\ \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}\frac{dy}{dx} + \frac{\partial F_2}{\partial z}\frac{dz}{dx} = 0 \end{cases}$$

$$\begin{bmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \end{bmatrix} + \begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \end{bmatrix}$$

If $\underline{\underline{\det \begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix} \neq 0}}$ at (a,b,c), i.e. $\begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix}$ is <u>invertible</u>,

then
$$\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = -\begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \end{bmatrix} \quad \text{at } (a,b,c)$$

General dimensions

Given $n+k$ variables, k equations

$(x_1, \cdots x_n, y_1, \cdots, y_k)$ $n+k$ variables

$$\begin{cases} F_1(x_1, \cdots, x_n, y_1, \cdots, y_k) = c_1 \\ \vdots \\ F_k(x_1, \cdots, x_n, y_1, \cdots, y_k) = c_k \end{cases}$$

expect: $y_1, \cdots, y_k$ can be solved as functions of $x_1, \cdots, x_n$.

(Ex: Try to write down the system from implicit differentiation)