Further examples

eg1 Find the point on the ellipse

$$x^2+xy+y^2=9 \qquad \text{(Check: it is really a ellipse!)}$$

with maximum x-coordinate.

$$\boxed{\begin{aligned} & x^2+xy+y^2=9 \\ \Leftrightarrow\ & \left(x^2+xy+\tfrac{1}{4}y^2\right)+\tfrac{3}{4}y^2=9 \\ \Leftrightarrow\ & \left(x+\tfrac{1}{2}y\right)^2+\tfrac{3}{4}y^2=9 \end{aligned}}$$

Soln: Let $f(x,y)=x$

& $g(x,y)=x^2+xy+y^2$

Maximize f under the constraint

$g=9$.

Consider $F(x,y,\lambda)=x-\lambda(x^2+xy+y^2-9)$

$$\begin{cases} 0=\dfrac{\partial F}{\partial x}=1-\lambda(2x+y) \\ 0=\dfrac{\partial F}{\partial y}=-\lambda(x+2y) \\ 0=\dfrac{\partial F}{\partial \lambda}=-(x^2+xy+y^2-9) \end{cases} \Leftrightarrow \begin{cases} \lambda(2x+y)=1 & \text{--- (1)} \\ \lambda(x+2y)=0 & \text{--- (2)} \\ x^2+xy+y^2=9 & \text{--- (3)} \end{cases}$$

$(1) \Rightarrow \lambda \neq 0$

Then $(2) \Rightarrow \quad x+2y=0 \Rightarrow x=-2y$

Sub. into (3), $\quad (-2y)^2+(-2y)y+y^2=9$

$$\Rightarrow \quad y=\pm\sqrt{3}$$

Hence $(x,y)=(-2\sqrt{3},\sqrt{3}),\ (2\sqrt{3},-\sqrt{3})$ are the critical points. Comparing values: $2\sqrt{3}>-2\sqrt{3}$

$\Rightarrow$ max value of x-coordinate is $2\sqrt{3}$

(at the point $(2\sqrt{3},-\sqrt{3})$) ※

eg 2 Find the point(s) on the hyperboloid

$$xy - yz - zx = 3$$

(Check: it is really a hyperboloid) ← of 2 sheets

closest to the origin

$$\boxed{\begin{aligned} & xy - yz - zx = 3 \\ \Leftrightarrow \quad & y(x-z) + \left(\frac{x-z}{2}\right)^2 - \left(\frac{x+z}{2}\right)^2 = 3 \\ \Leftrightarrow \quad & \left(y^2 + 2y\left(\frac{x-z}{2}\right) + \left(\frac{x-z}{2}\right)^2\right) - y^2 - \left(\frac{x+z}{2}\right)^2 = 3 \\ \Leftrightarrow \quad & \left(y + \frac{x-z}{2}\right)^2 - y^2 - \left(\frac{x+z}{2}\right)^2 = 3 \end{aligned}}$$

Soln:

Let $f(x,y,z) = x^2+y^2+z^2$

$g(x,y,z) = xy - yz - zx$

Minimize f under constraint $g = 3$.

Consider $F(x,y,z,\lambda) = x^2+y^2+z^2 - \lambda(xy-yz-zx-3)$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x} = 2x - \lambda(y-z) & \text{——— (1)} \\ 0 = \dfrac{\partial F}{\partial y} = 2y - \lambda(x-z) & \text{——— (2)} \\ 0 = \dfrac{\partial F}{\partial z} = 2z + \lambda(x+y) & \text{——— (3)} \\ 0 = \dfrac{\partial F}{\partial \lambda} = -(xy-yz-zx-3) & \text{——— (4)} \end{cases}$$

Case 1 $\lambda = 0$,

then (1), (2) & (3) $\Rightarrow$ $x = y = z = 0$

contradicting eqt. (4)

Case 2 $\lambda \neq 0$

Then (1), (2) & (3) $\Rightarrow$

$$\begin{cases} y-z = \frac{2}{\lambda} x & \text{——— (5)} \\ x-z = \frac{2}{\lambda} y & \text{——— (6)} \\ x+y = -\frac{2}{\lambda} z & \text{——— (7)} \end{cases}$$

$(5)-(6) \Rightarrow \quad y-x = \frac{2}{\lambda}(x-y) \Rightarrow \left(1+\frac{2}{\lambda}\right)(x-y)=0$ ——— (8)

$(7)-(6) \Rightarrow \quad y+z = -\frac{2}{\lambda}(z+y) \Rightarrow \left(1+\frac{2}{\lambda}\right)(y+z)=0$ — (9)

Subcase (i) $\quad 1+\frac{2}{\lambda}=0$

i.e. $\quad \lambda=-2$

then $(5)+(6)+(7) \Rightarrow \quad x+y-z=0$

$$\Rightarrow \quad 0=(x+y-z)^2 = x^2+y^2+z^2+2(xy-yz-xz)$$

$$= x^2+y^2+z^2+6 \qquad (\text{by } (4))$$

which is a contradiction.

Subcase (ii) $\quad 1+\frac{2}{\lambda} \neq 0$

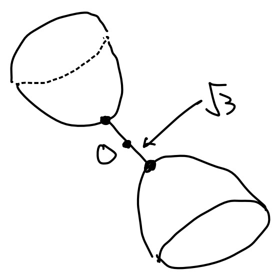
Then by (8) & (9), $\quad x=y=-z$

Sub. into (4) $\quad x^2+x^2+x^2=3 \Rightarrow x^2=1$

$\Rightarrow \quad x=\pm 1$

$\Rightarrow \quad (x,y,z)=(\pm 1, \pm 1, \mp 1) = \pm(1,1,-1)$

$f(1,1,-1)=f(-1,-1,1)=3$



$\Rightarrow$ closest points are $\pm(1,1,-1)$

with corresponding distance $=\sqrt{3}$

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Lagrange Multipliers with multiple Constraints

Let
- $f, g_1, \cdots, g_k : \Omega \to \mathbb{R}$ be C^1 functions, ($\Omega \subseteq \mathbb{R}^n$, open)
- $S = \{ \vec{x} \in \Omega : g_i(\vec{x}) = c_i$ for $i = 1, \cdots, k \}$

Suppose
- $\vec{a}$ is a local extremum of f on S
- $\vec{\nabla} g_1(\vec{a}), \cdots, \vec{\nabla} g_k(\vec{a})$ are linearly independent vectors

Then
$$\begin{cases} \vec{\nabla} f(\vec{a}) = \sum_{i=1}^{k} \lambda_i \vec{\nabla} g_i(\vec{a}) \\ g_i(\vec{a}) = c_i \ , \ i = 1, \cdots, k \end{cases}$$

for some Lagrange multipliers $\lambda_1, \cdots, \lambda_k \in \mathbb{R}$.

Same as 1 constraint,

Finding extrema of $f(\vec{x})$ with constraints $g_i(\vec{x}) = c_i$, $i = 1, \cdots, k$

$$\Updownarrow$$

Finding extrema of $F(\vec{x}, \lambda_1, \cdots, \lambda_k) = f(\vec{x}) - \sum_{i=1}^{k} \lambda_i (g_i(\vec{x}) - c_i)$

without constraint

(but more variables : adding λ_i as new variables) eg1 Maximize $f(x,y,z) = x^2 + 2y - z^2$

on the line $L: \begin{cases} 2x - y = 0 \\ y + z = 0 \end{cases}$ in $\mathbb{R}^3$

(Given that maximum exists)

Soln Let $g_1(x,y,z) = 2x - y$

$g_2(x,y,z) = y + z$

Maximize f subject to constraints $\begin{cases} g_1 = 0 \\ g_2 = 0 \end{cases}$

$\left[\begin{array}{l} f \text{ is degree 2 poly.} \\ g_1 \,\&\, g_2 \text{ are degree 1 poly.} \end{array} \right\} \Rightarrow f, g_1, g_2 \text{ are } C^1 \Big]$

Consider

$$F(x,y,z,\lambda_1,\lambda_2) = x^2 + 2y - z^2 - \lambda_1(2x - y) - \lambda_2(y + z)$$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x} = 2x - 2\lambda_1 \\ 0 = \dfrac{\partial F}{\partial y} = 2 + \lambda_1 - \lambda_2 \\ 0 = \dfrac{\partial F}{\partial z} = -2z - \lambda_2 \\ 0 = \dfrac{\partial F}{\partial \lambda_1} = -(2x - y) \\ 0 = \dfrac{\partial F}{\partial \lambda_2} = -(y + z) \end{cases} \Leftrightarrow \begin{cases} x = \lambda_1 & \text{——— } (1) \\ \lambda_2 = \lambda_1 + 2 & \text{——— } (2) \\ \lambda_2 = -2z & \text{——— } (3) \\ 2x = y & \text{——— } (4) \\ y = -z & \text{——— } (5) \end{cases}$$

(1) & (3) sub. into (2) $\Rightarrow$ $-2z = x + 2$ ——— (6)

(4) & (5) $\Rightarrow$ $2x = y = -z$ ——— (7)

Sub. into (6) $4x = x + 2 \Rightarrow x = \frac{2}{3}$

Sub. into (7) $\Rightarrow$ $y = \frac{4}{3}$, $z = -\frac{4}{3}$

$\Rightarrow$ max occurs at $\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$

with value $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \left(\frac{2}{3}\right)^2 + 2\left(\frac{4}{3}\right) - \left(\frac{4}{3}\right)^2 = \frac{4}{3}$

(Check!)

$$\left[\begin{array}{l} \vec{\nabla} g_1 = (2, -1, 0) \\ \vec{\nabla} g_2 = (0, 1, 1) \end{array}\right\} \text{ are linearly independent} \Bigg]$$

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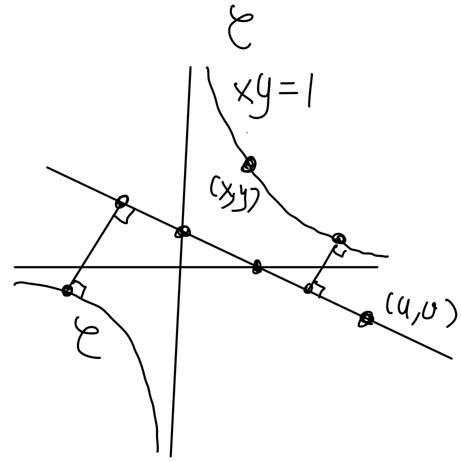
eg2 Find the distance between the hyperbola $\mathcal{E}: xy = 1$ and the line $L: x + 4y = \frac{15}{8}$



Soln: Let $f(x,y,u,v)$

$$= (x-u)^2 + (y-v)^2$$

minimize f under constraints

$$\begin{cases} g_1(x,y,u,v) = xy = 1 \\ g_2(x,y,u,v) = u + 4v = \frac{15}{8} \end{cases}$$

$$\left[\begin{array}{l} \vec{\nabla} g_1 = \left(\frac{\partial g_1}{\partial x}, \frac{\partial g_1}{\partial y}, \frac{\partial g_1}{\partial u}, \frac{\partial g_1}{\partial v}\right) = (y, x, 0, 0) \\ \vec{\nabla} g_2 = \left(\frac{\partial g_2}{\partial x}, \frac{\partial g_2}{\partial y}, \frac{\partial g_2}{\partial u}, \frac{\partial g_2}{\partial v}\right) = (0, 0, 1, 4) \end{array} \quad \begin{array}{l} \text{linearly indep.} \\ (\text{for } (x,y) \neq (0,0)) \end{array}\right]$$

Consider $F(x,y,z,\lambda_1,\lambda_2) = (x-u)^2+(y-v)^2-\lambda_1(xy-1)-\lambda_2\left(u+4v-\frac{15}{8}\right)$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x} = 2(x-u) - \lambda_1 y & \text{———— (1)} \\ 0 = \dfrac{\partial F}{\partial y} = 2(y-v) - \lambda_1 x & \text{———— (2)} \\ 0 = \dfrac{\partial F}{\partial u} = -2(x-u) - \lambda_2 & \text{———— (3)} \\ 0 = \dfrac{\partial F}{\partial v} = -2(y-v) - 4\lambda_2 & \text{———— (4)} \\ 0 = \dfrac{\partial F}{\partial \lambda_1} = -(xy-1) & \text{———— (5)} \\ 0 = \dfrac{\partial F}{\partial \lambda_2} = -\left(u+4v-\frac{15}{8}\right) & \text{———— (6)} \end{cases}$$

Case 1 If $\lambda_1 = 0$ or $\lambda_2 = 0$, $x=u$ & $y=v$

Sub. into (6) $\Rightarrow$ $x = \frac{15}{8} - 4y$

Sub. into (5) $\Rightarrow$ $\left(\frac{15}{8} - 4y\right)y = 1$

$4y^2 - \frac{15}{8}y + 1 = 0$ has no (real) solution.

Case 2 $\lambda_1 \neq 0$ & $\lambda_2 \neq 0$

Then (3) & (4) $\dfrac{x-u}{y-v} = \dfrac{1}{4}$

& (1) & (2) $\dfrac{x-u}{y-v} = \dfrac{y}{x}$

$\Rightarrow$ $x = 4y$

Sub. into (5) $(4y)y = 1 \Rightarrow y = \pm\frac{1}{2}$

$\therefore$ $(x,y) = \pm\left(2, \frac{1}{2}\right)$ $(\neq (0,0))$

Then for $(2, \frac{1}{2})$, $\frac{2-u}{\frac{1}{2}-v} = \frac{1}{4} \Rightarrow 4u - v = \frac{15}{2}$

together $u + 4v = \frac{15}{8}$

$\Rightarrow (u, v) = (\frac{15}{8}, 0)$ (check!)

Similarly for $(-2, -\frac{1}{2})$, we have $(u, v) = (-\frac{225}{136}, \frac{15}{17})$ (Ex!)

Comparing values: $f(2, \frac{1}{2}, \frac{15}{8}, 0) = \frac{17}{64}$ (check!)

$f(-2, -\frac{1}{2}, -\frac{225}{136}, \frac{15}{17}) = \cdots > \frac{17}{64}$ (check!)

$\Rightarrow$ distance between $\mathcal{E}$ and $L = \frac{\sqrt{17}}{8}$

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