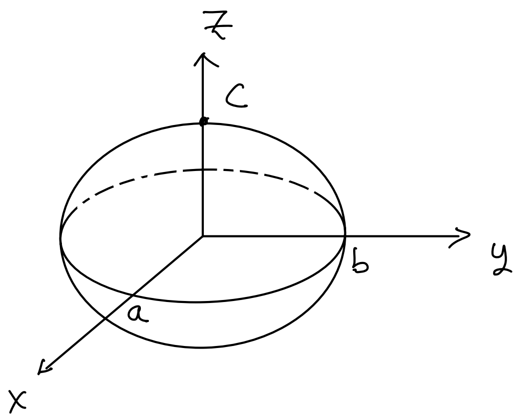
Quadratic Constraint for 3-variables

$$g(x,y,z) = Ax^2 + By^2 + Cz^2 + 2Pxy + 2Qyz + 2Rzx + Dx + Ey + Fz + G$$

Some typical examples of $g = \text{const.}$

eg 1 $\qquad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Ellipsoid



eg2 $\qquad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ $\quad$ Hyperboloid of 1 sheet

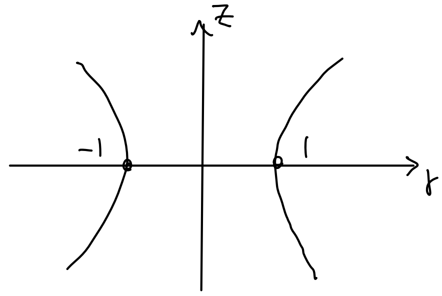
Up to scaling of each variables, graph looks like the graph of

$$x^2 + y^2 - z^2 = 1$$

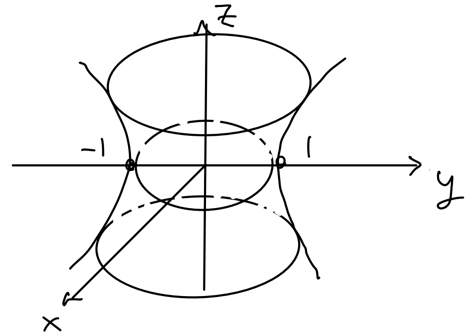
Using cylindrical coordinates : $\left\{\begin{array}{l} \left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \text{polar} \\ z = z \end{array}\right.$

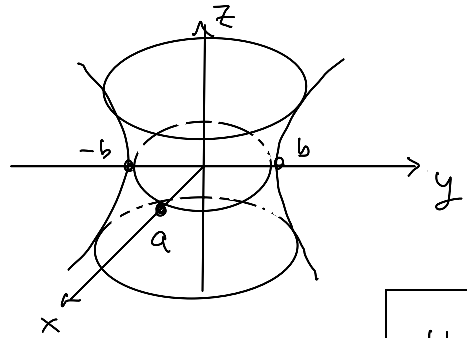
then the constraint can be written as

$$r^2 - z^2 = 1 \qquad (\text{since } r^2 = x^2 + y^2)$$



same for all direction
$\xrightarrow[\theta]{}$





scaling back to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

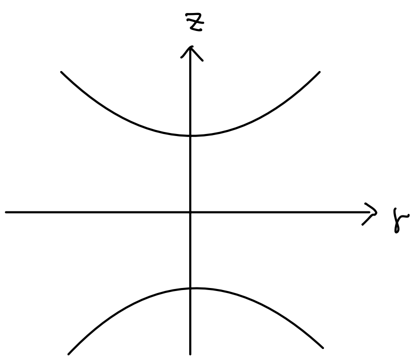
$\boxed{\text{Hyperboloid of 1 sheet}}$

eg3 $\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=-1 \qquad$ Hyperboloid of 2 sheets

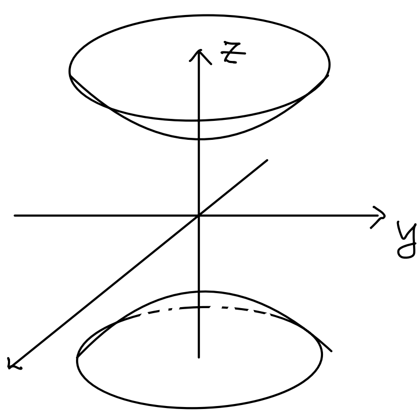
Similarly, after scaling, looks like

$$x^2+y^2-z^2=-1 \iff r^2-z^2=-1 \text{ (cylindrical coordinates)}$$

$$\iff z^2-r^2=1$$



same for all direction
θ
$\longrightarrow$



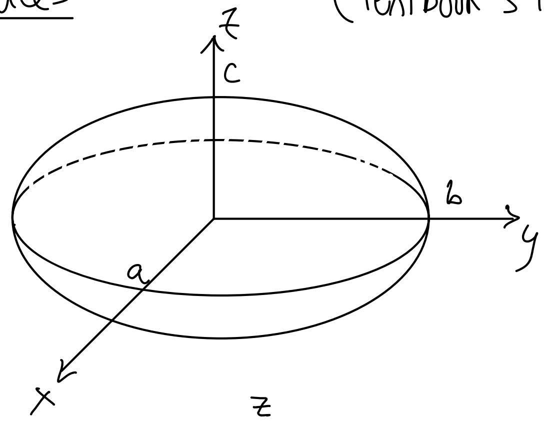
$\boxed{\text{Hyperboloid of 2 sheets}}$

In summary, we have

Graphs of Standard Quadratic Surfaces (Textbook §11.6)

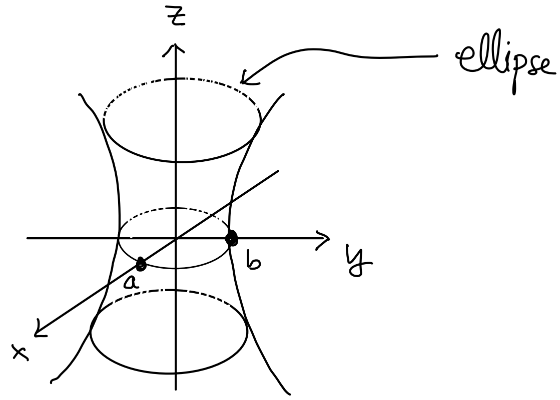
Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



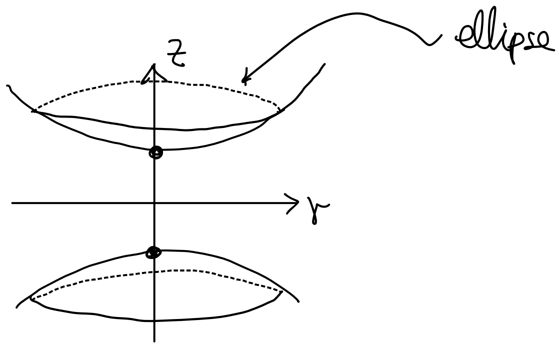
Hyperboloid of 1 sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



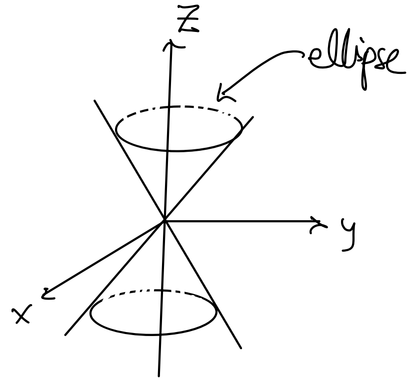
Hyperboloid of 2 sheets

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$



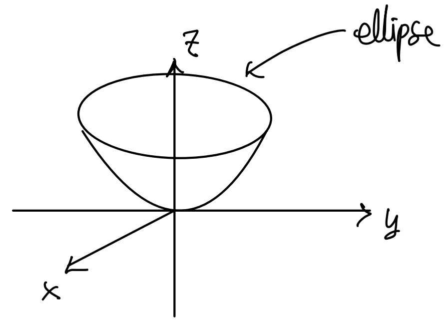
Elliptic Cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$



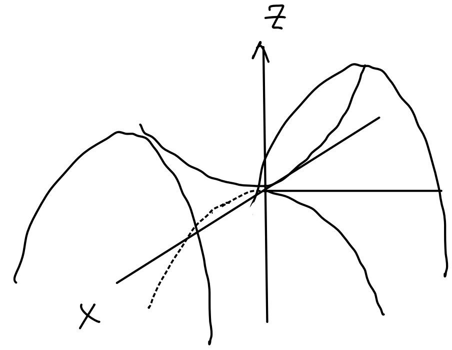
Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



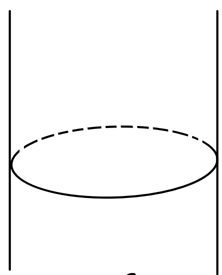
Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

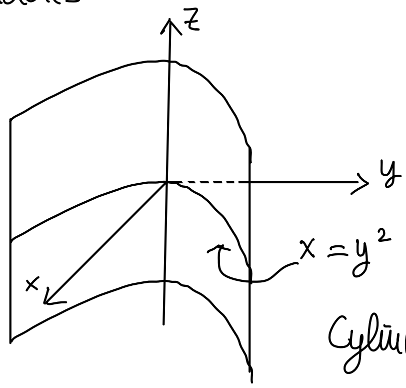


Degenerate to cylinders over conic sections

eg: Equation involve NO z variables



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cylinder over ellipse



and etc.

$x = y^2$

Cylinder over parabola

Other degenerate cases

eg $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0 \quad \& \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$

> Fact Any quadratic constraint $g(x,y,z) = c$ can be transformed to one of the standard forms by a change of coordinates.

Remark: As in 2-variables, only ellipsoid is closed and bounded