Higher dimension example

eg: $q(x,y,z) = xy + yz + zx$

has definite sign for $(x,y,z) \neq (0,0,0)$?

Answer: NO

Soln: Trick: $q(x,y,z) = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2 + z(x+y)$

Let $u = \frac{x+y}{2}$, $v = \frac{x-y}{2}$

then
$$
\begin{aligned}
q &= u^2 - v^2 + 2zu \\
&= (u^2 + 2uz + z^2) - z^2 - v^2 \\
&= (u+z)^2 - z^2 - v^2 \\
&= \frac{1}{4}(x+y+2z)^2 - \frac{1}{4}(x-y)^2 - z^2
\end{aligned}
$$

On the plane $x+y+2z = 0$ (ie. $z = -\frac{x+y}{2}$),

$$q = q\left(x, y, -\frac{x+y}{2}\right) = -\frac{1}{4}(x-y)^2 - \frac{1}{4}(x+y)^2$$

$$< 0 \quad \text{for } \left(x, y, -\frac{x+y}{2}\right) \neq \vec{0}$$

Along the line $\begin{cases} x-y=0 \\ z=0 \end{cases} \Rightarrow \begin{cases} x=y \\ z=0 \end{cases}$

$$q = q(x,x,0) = \frac{1}{4}(x+x+0)^2 - 0^2 - 0^2 = x^2 > 0,$$

$\forall\, x \neq 0$, ie $(x,x,0) \neq \vec{0}$

(Together $\Rightarrow$ $(0,0,0)$ is a saddle point) ※

Second Derivative Test for general n

Recall f is $C^2 \Rightarrow$ (by Clairaunt's / mixed derivative Thm)

$$Hf(\vec{a}) = \left[f_{x_i x_j} \right]_{i,j=1,\cdots,n} \quad \text{is symmetric}$$

Theory of Linear Algebra $\Rightarrow$ Hf is diagonizable

i.e. $\exists$ orthogonal $n \times n$ matrix P (ie $P^T P = Id$) s.t.

$$P^T Hf(\vec{a}) P = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

where λ_i, $i = 1, \cdots, n$, are eigenvalues of $Hf(\vec{a})$.

$$\Rightarrow \quad \boxed{Hf(\vec{a}) \text{ is } \begin{cases} \text{positive definite} \Leftrightarrow \text{all } \lambda_i > 0 \\ \text{negative definite} \Leftrightarrow \text{all } \lambda_i < 0 \\ \text{indefinite} \Leftrightarrow \text{some } \lambda_i > 0, \text{ some } \lambda_j < 0 \text{ (all } \neq 0) \end{cases}}$$

Another way to check is consider determinants of submatrix

For each $1 \leq k \leq n$,

consider submatrix H_k given by

the upper left $k \times k$ entries.

$$\begin{bmatrix} f_{x_1 x_1} & \cdots & f_{x_1 x_k} & \cdots & f_{x_1 x_n} \\ \vdots & & \vdots & & \vdots \\ f_{x_k x_1} & \cdots & f_{x_k x_k} & \cdots & f_{x_k x_n} \\ \vdots & & \vdots & & \vdots \\ f_{x_n x_1} & \cdots & f_{x_n x_k} & \cdots & f_{x_n x_n} \end{bmatrix}$$

Then

$$\boxed{\begin{array}{l} Hf(\vec{a}) \text{ is positive definite} \iff \det H_k > 0, \ \forall k = 1, \cdots, n \\ Hf(\vec{a}) \text{ is negative definite} \iff \det H_k \begin{cases} < 0, & k \text{ odd} \\ > 0, & k \text{ even} \end{cases} \end{array}}$$

<u>egs</u> (1) $n=2$ $\quad \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ has $\quad H_1 = [f_{xx}]$

$$H_2 = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$\Rightarrow \quad \det H_1 = f_{xx}$$

$$\det H_2 = f_{xx} f_{yy} - f_{xy}^2$$

(Same result as before)

(2) Diagonal matrix $\begin{bmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_k & & \\ & & & \ddots & \\ 0 & & & & \lambda_n \end{bmatrix} \Rightarrow H_k = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{bmatrix}$

$$\Rightarrow \det H_k = \lambda_1 \cdots \lambda_k .$$

Lagrange Multipliers

(A method for <u>finding extrema under constraints</u>)

<u>eg1</u> In previous example of finding global max/min of $f(x,y) = x^2+2y^2-x+3$ for $x^2+y^2 \leq 1$, one need to find (in step 2) the max/min values of f on the boundary $x^2+y^2=1$.

In otherwords, finding global max/min (on $x^2+y^2=1$)

of $\qquad f(x,y) = x^2+2y^2-x+3$

under constraint $\quad g(x,y) = x^2+y^2 = 1$

Another typical example :

<u>eg2</u> Find the point on the parabola $x^2=4y$ closest to $(1,2)$.

i.e. Find (global) minimum of

$$f(x,y) = (x-1)^2+(y-2)^2$$

(equivalent to, but easier than $\sqrt{(x-1)^2+(y-2)^2}$)

under constraint

$$g(x,y) = x^2-4y = 0 .$$

<u>Remark</u>: In both examples, constraints are expressed as level set. $g=c$ for some constant c.

Thm (Lagrange Multipliers)

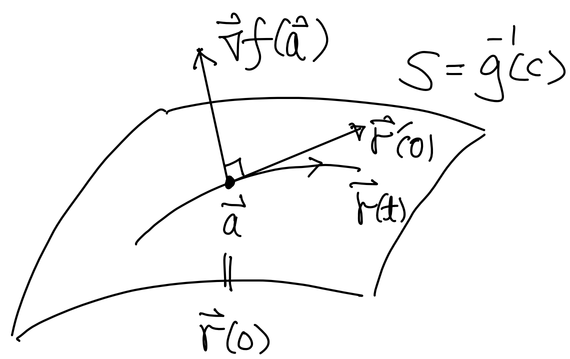
Let
- $f, g : \Omega \to \mathbb{R}$ be C^1 functions, ($\Omega \subset \mathbb{R}^n$ open)
- $S = g^{-1}(c) = \{x \in \Omega : g(x) = c\}$ be a level set of g

Suppose
- $\vec{a} \in S$ is a local extremum of f restricted to S (i.e. under the constraint $g = c$)
- $\vec{\nabla} g(\vec{a}) \neq \vec{0}$

Then
- $\vec{\nabla} f(\vec{a}) = \lambda \vec{\nabla} g(\vec{a})$ for some $\lambda \in \mathbb{R}$
- $g(\vec{a}) = c$

where λ is called a Lagrange Multiplier

(Pf: Omitted)



Idea: $f(\vec{r}(t))$ has an local extreme at $t=0$, corresponding to $\vec{a} = \vec{r}(0)$

$$f(\vec{r}(0)) = f(\vec{a}) \underset{(\le)}{\ge} f(\vec{r}(t)), \ \forall t.$$

$$\Rightarrow \quad 0 = \left.\frac{d}{dt}\right|_{t=0} f(\vec{r}(t)) = \vec{\nabla} f(\vec{r}(0)) \cdot \vec{r}\,'(0)$$

Since it is true for all curves on S passing thro. $\vec{a}$,

$\Rightarrow$ $\vec{\nabla} f(\vec{a})$ perpendicular to $S = g^{-1}(c)$ at $\vec{a}$.

$\Rightarrow$ $\vec{\nabla} f(\vec{a})$ is normal to S $\Rightarrow$ $\vec{\nabla} f(\vec{a}) \,//\, \vec{\nabla} g(\vec{a})$

$\therefore$ $\vec{\nabla} f(\vec{a}) = \lambda \vec{\nabla} g(\vec{a})$ for some $\lambda \in \mathbb{R}$

$\#$

<u>Reduction to unconstrainted problem</u> (By Lagrange Multiplier)

> Finding extrema of $f(\vec{x})$ with constraint $g(\vec{x}) = c$
>
> $\Updownarrow$
>
> Finding extrema of $F(\vec{x}, \lambda) = f(\vec{x}) - \lambda(g(\vec{x}) - c)$
>
> without constraint
>
> (but more variables : adding λ as a new variable)

<u>Idea</u>: (To be cont'd)