

Application to local max/min

If $f \in C^2$, and \vec{a} is a critical point of f .

Then $\vec{\nabla} f(\vec{a}) = \vec{0}$

$$\Rightarrow f(\vec{x}) \approx P_2(\vec{x}) = f(\vec{a}) + \vec{\nabla} f(\vec{a})(\vec{x} - \vec{a}) + \frac{1}{2}(\vec{x} - \vec{a})^T H f(\vec{a})(\vec{x} - \vec{a})$$

$$f(\vec{x}) - f(\vec{a}) \approx \frac{1}{2}(\vec{x} - \vec{a})^T H f(\vec{a})(\vec{x} - \vec{a})$$

\therefore If $(\vec{x} - \vec{a})^T H f(\vec{a})(\vec{x} - \vec{a}) < 0 \quad \forall \vec{x} \text{ near } \vec{a}$ (but $\vec{x} \neq \vec{a}$)

then $f(\vec{x}) < f(\vec{a}) \quad \forall \vec{x} \text{ near } \vec{a}$

$\Rightarrow \vec{a}$ is a local max

If $(\vec{x} - \vec{a})^T H f(\vec{a})(\vec{x} - \vec{a}) > 0 \quad \forall \vec{x} \text{ near } \vec{a}$ (but $\vec{x} \neq \vec{a}$)

then $f(\vec{x}) > f(\vec{a}) \quad \forall \vec{x} \text{ near } \vec{a}$

$\Rightarrow \vec{a}$ is a local min

So we need to study when is a sym matrix H satisfies

$$\vec{v}^T H \vec{v} > 0 \quad \forall \text{ vector } \vec{v} \neq \vec{0}$$

$$\text{and } \vec{v}^T H \vec{v} < 0 \quad \forall \text{ vector } \vec{v} \neq \vec{0}$$

Hence we make the following

Dof: Let H be a symmetric $n \times n$ matrix.

Then H is said to be

(1) positive definite if $\vec{x}^T H \vec{x} > 0$

for all column vectors $\vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$

(2) negative definite if $\vec{x}^T H \vec{x} < 0$

for all column vectors $\vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$

(3) indefinite if \exists column vectors $\vec{x}, \vec{y} \in \mathbb{R}^n \setminus \{\vec{0}\}$

such that $\vec{x}^T H \vec{x} > 0$ and $\vec{y}^T H \vec{y} < 0$

Remark: These are not all possibilities: \exists sym. matrix which is not positive definite, negative definite, nor indefinite.

Then the discussion above implies

Thm (Second Derivative Test)

Let $\begin{cases} \bullet f: \mathcal{D} \rightarrow \mathbb{R} \text{ be } C^2, \mathcal{D} \subseteq \mathbb{R}^n, \text{ open} \\ \bullet \vec{a} \in \mathcal{D} \text{ such that } \vec{\nabla} f(\vec{a}) = \vec{0} \end{cases}$

Then

$Hf(\vec{a})$ is $\begin{cases} \text{positive definite} \Rightarrow \vec{a} \text{ is a local min} \\ \text{negative definite} \Rightarrow \vec{a} \text{ is a local max} \\ \text{indefinite} \Rightarrow \vec{a} \text{ is a saddle point} \end{cases}$

Remark: A critical point which is neither local max nor local min is called a saddle point.

In particular for 2-variable, $\vec{v}^T H \vec{v}$ is of the form

$$g(x, y) = [x \ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bx\bar{y} + cy^2$$

(1) $[x, y] \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 4y^2 > 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ is positive definite

(2) $[x, y] \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -x^2 - 4y^2 < 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$

$\therefore \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$ is negative definite

(3) $[x, y] \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -x^2 + 4y^2$

If $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $[x, y] \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 4 > 0$

If $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $[x, y] \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -1 < 0$

$\therefore \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ is indefinite

(4) $[x, y] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 \geq 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$

But $\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 (\neq 0) \Rightarrow$ not positive definite

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not positive definite, negative definite, nor indefinite.

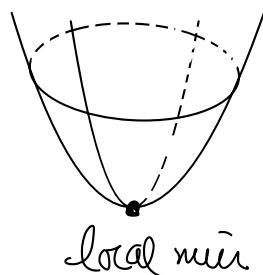
$$\begin{aligned}
 (5) \quad [x, y] \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= x^2 + 4xy + 5y^2 \\
 &= (x^2 + 4xy + 4y^2) + y^2 \\
 &= (x+2y)^2 + y^2 \\
 &> 0 \quad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}
 \end{aligned}$$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ is positive definite. \times

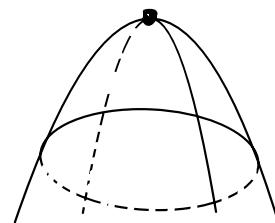
Geometrically (locally near the critical point)

(1) $Hf(\vec{a})$ positive definite

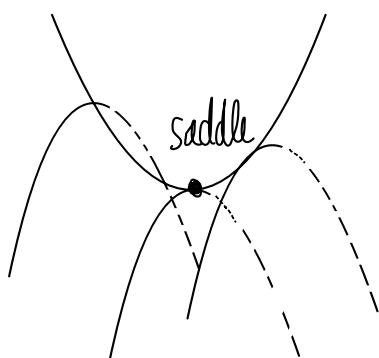
The graph of $z = [x \ y] Hf(\vec{a}) \begin{bmatrix} x \\ y \end{bmatrix}$ looks like



(2) $Hf(\vec{a})$ negative definite local max



(3) $Hf(\vec{a})$ is indefinite



Then let $H = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

Then

H is $\begin{cases} \text{positive definite} \Leftrightarrow \det H = ac - b^2 > 0, \& a > 0 \\ \text{negative definite} \Leftrightarrow \det H = ac - b^2 > 0, \& a < 0 \\ \text{indefinite} \Leftrightarrow \det H = ac - b^2 < 0 \end{cases}$

$$\text{Pf: If } a \neq 0, [x \ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bx + cy^2$$

$$= a(x^2 + 2\frac{b}{a}xy) + cy^2 = a\left(x + \frac{b}{a}y\right)^2 + \frac{ac - b^2}{a^2}y^2$$

$$(1) \det H = ac - b^2 > 0 \Leftrightarrow$$

$[x \ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ has the same sign as a

\therefore The 1st two statements are proved.

$$(2) \det H = ac - b^2 < 0 \Leftrightarrow$$

2 terms have different sign

$\Leftrightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is indefinite

$$\text{If } a=0 \Rightarrow \det H = ac - b^2 = -b^2 \leq 0$$

$$[x \ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2bx + cy^2$$

If $b=0$, then $[x \ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = cy^2$ not positive,
not negative,
nor indefinite
($\downarrow \det H = 0$)

$$\text{If } b \neq 0, \text{ then } zbx^2 + cy^2$$

\Downarrow

$$(\det H < 0) = zby(x + \frac{c}{zb}y)$$

$$= \frac{b}{z} \left[\left(y + \left(x + \frac{c}{zb}y \right) \right)^2 - \left(y - \left(x + \frac{c}{zb}y \right) \right)^2 \right]$$

\Rightarrow also indefinite (using $4uv = (u+v)^2 - (u-v)^2$)

Still in the case of $\det H = -b^2 < 0 \Leftrightarrow$ indefinite ~~indefinite~~

eg 1 $g(x,y) = zxy = [x \ y] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \left(= \frac{1}{2}(x+y)^2 - \frac{1}{2}(x-y)^2 \right)$

$$\det H = -1 < 0 \Rightarrow \text{indefinite}$$

$(a=0)$

eg 2: $g(x,y) = 17x^2 - 12xy + 8y^2 = [x \ y] \begin{bmatrix} 17 & -6 \\ -6 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$\det \begin{bmatrix} 17 & -6 \\ -6 & 8 \end{bmatrix} = 17 \cdot 8 - (-6)^2 = 100 > 0 \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow \begin{array}{l} \text{positive} \\ \text{definite} \end{array}$$

$a = 17 > 0$

$$\left(g(x,y) = 17 \left(x - \frac{6}{17}y \right)^2 + \frac{100}{17}y^2 \right)$$

Then for 2-variables, the 2nd derivative test is

Thm (Second Derivative Test for 2-variables)

Let $\begin{cases} \bullet f: \Omega \rightarrow \mathbb{R} \text{ be } C^2, \Omega \subseteq \mathbb{R}^2, \text{ open} \\ \bullet (a,b) \in \Omega \text{ such that } \vec{\nabla} f(a,b) = \vec{0} \end{cases}$

Then

(1) $f_{xx}f_{yy} - f_{xy}^2 > 0$ & $f_{xx} > 0$ at $(a,b) \Rightarrow (a,b)$ is a local min

(2) $f_{xx}f_{yy} - f_{xy}^2 > 0$ & $f_{xx} < 0$ at $(a,b) \Rightarrow (a,b)$ is a local max

(3) $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a,b) \Rightarrow (a,b)$ is a saddle point

(4) $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a,b) \Rightarrow$ inconclusive.

Remark $f_{xx}f_{yy} - f_{xy}^2 = \det Hf$ (for 2-variables)

e.g (1) $f(x,y) = x^3 + y^2$

$$f_{xx} = 6x, \quad f_{xy} = 0, \quad f_{yy} = 2$$

$$Hf = \begin{bmatrix} 6x & 0 \\ 0 & 2 \end{bmatrix} \quad \det Hf = 12x$$

$$\vec{\nabla} f = \vec{0} \Leftrightarrow (3x^2, 2y) = (0, 0) \Leftrightarrow (x, y) = (0, 0)$$

$$\det Hf(0,0) = 0$$

$f(t,0) > 0, \quad f(-t,0) < 0$ for all $t > 0$ small

$\Rightarrow (0,0)$ is a saddle point.

$$(2) \quad g(x,y) = x^4 + y^4 \quad \& \quad h(x,y) = -x^4 - y^4$$

Clearly $(0,0)$ is a critical point of both g & h

$$Hg = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix} \quad Hh = \begin{bmatrix} -12x^2 & 0 \\ 0 & -12y^2 \end{bmatrix}$$

$$\Rightarrow \det Hg(0,0) = 0 = \det Hh(0,0)$$

However, $(0,0)$ is a minimum pt. of g ,
 $(0,0)$ is a maximum pt. of h .

(1) & (2) \Rightarrow critical point (a,b) , can be local max
local min, or saddle in the case
when $\det Hf(a,b) = 0$.

Q1 $f(x,y) = 3x^2 - 10xy + 3y^2 + 2x + 2y + 3$

Find and classify critical points of f .

Soh : (f polynomial always C^2)

$$\vec{\nabla} f = [6x - 10y + 2, \quad -10x + 6y + 2]$$

$$Hf = \begin{bmatrix} 6 & -10 \\ -10 & 6 \end{bmatrix} \quad (\text{a constant matrix})$$

$$\text{Critical pts: } \vec{\nabla} = \vec{\nabla} f \Leftrightarrow \begin{cases} 6x - 10y + 2 = 0 \\ -10x + 6y + 2 = 0 \end{cases}$$

$$\stackrel{\text{check}}{\Leftrightarrow} (x, y) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\det Hf = f_{xx}f_{yy} - f_{xy}^2 = 6 \cdot 6 - (-10)^2 = -64 < 0$$

$\Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right)$ is a saddle point (by 2nd derivative test)
 (No need to check $f_{xx} = 6 > 0$) ~~XX~~

$$\text{Q2: } f(x, y) = 3x - x^3 - 3xy^2$$

Find and classify critical points of f

Solu (f polynomial always C^2)

$$\vec{\nabla} f = [3 - 3x^2 - 3y^2, -6xy]$$

$$Hf = \begin{bmatrix} -6x & -6y \\ -6y & -6x \end{bmatrix}$$

$$\text{Critical pts: } \vec{\nabla} = \vec{\nabla} f \Leftrightarrow \begin{cases} 3 - 3x^2 - 3y^2 = 0 \\ -6xy = 0 \end{cases}$$

(check)

$$\Leftrightarrow (x, y) = (0, 1), (0, -1), (1, 0) \text{ or } (-1, 0)$$

(4 critical points)

Critical point	Hf	$f_{xx}f_{yy} - f_{xy}^2 = \det Hf$	f_{xx}	classification
(0,1)	$\begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$	$-36 < 0$	No need	saddle
(0,-1)	$\begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$	$-36 < 0$	No need	saddle
(1,0)	$\begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$	$36 > 0$	$-6 < 0$	local max
(-1,0)	$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$	$36 > 0$	$6 > 0$	local min

~~X~~

Q3 (Inconclusive from 2nd derivative test)

$$f(x,y) = x^2 + y^4, \quad g(x,y) = x^2 - y^4, \quad h(x,y) = -x^2 - y^4$$

(at (0,0): local min , saddle , local max)

Soh

$$\vec{\nabla}f = [2x, 4y^3], \quad \vec{\nabla}g = [2x, -4y^3], \quad \vec{\nabla}h = [-2x, -4y^3]$$

$$Hf = \begin{bmatrix} 2 & 0 \\ 0 & 12y^2 \end{bmatrix}, \quad Hg = \begin{bmatrix} 2 & 0 \\ 0 & -12y^2 \end{bmatrix}, \quad Hh = \begin{bmatrix} -2 & 0 \\ 0 & -12y^2 \end{bmatrix}$$

$$\det Hf(0,0) = \det Hg(0,0) = \det Hh(0,0) = 0$$

\therefore 2nd derivative test is inconclusive . ~~X~~