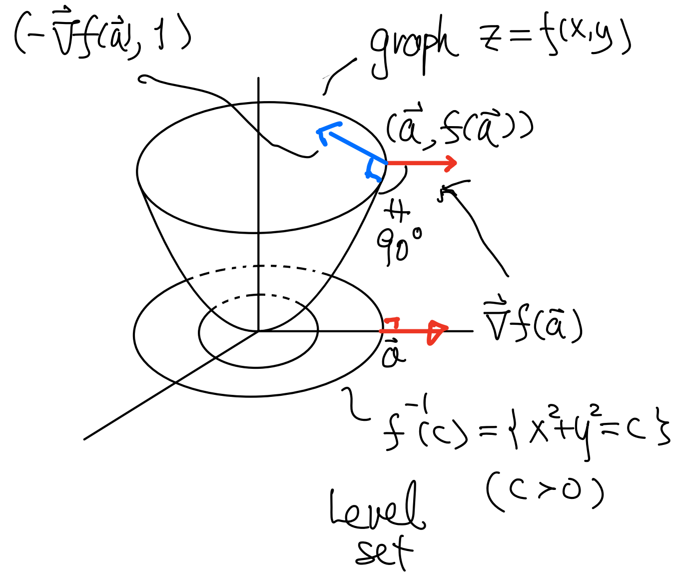
Remark (<u>Level set</u> vs <u>Graph</u>)



<u>eg</u> $f(x,y)=x^2+y^2$

graph $z=f(x,y)$

In general,

$\vec{\nabla}f(\vec{a})$ is normal to $f^{-1}(c)$ at $\vec{a}$ $(f(\vec{a})=c)$

but not normal to the graph of $z=f(x,y)$ at $(\vec{a}, f(\vec{a}))$

What is the normal of the graph of $z=f(x,y)$ at $(\vec{a}, f(\vec{a}))$?

Trick is : regard graph of $z=f(x,y)$ as 0-level set of $F(x,y,z)=z-f(x,y)$ at $(\vec{a}, f(\vec{a}))$ $\quad \left(F(\vec{a}, f(\vec{a})) = f(\vec{a})-f(\vec{a})=0\right)$

$\vec{a}=(x,y)$

Then by the Thm, the normal of the graph is

$$\vec{\nabla}F(\vec{a}, f(\vec{a})) = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) \quad (\text{at } (x,y)=\vec{a})$$

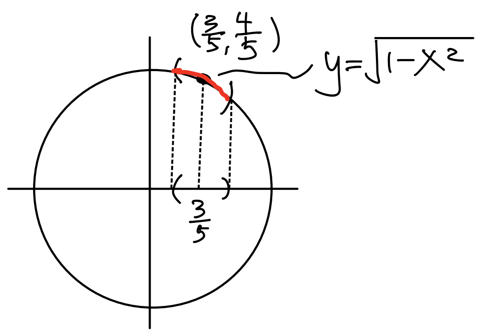
$$= (-\vec{\nabla}f(\vec{a}), 1) \quad \left(\begin{matrix}\text{1-dim'l} \\ \text{higher}\end{matrix}\right)$$

Another Application of Chain Rule:

Implicit Differentiation

eg 1 $C: x^2+y^2=1$ (y can be solved in term of x (for most x))

Find $\frac{dy}{dx}$ at $\left(\frac{3}{5}, \frac{4}{5}\right)$.



Soln: $x^2 + (y(x))^2 = 1$

near $(x,y) = \left(\frac{3}{5}, \frac{4}{5}\right)$

$\Rightarrow \quad \frac{d}{dx}\left(x^2+(y(x))^2\right) = \frac{d}{dx}1 = 0$

$\Rightarrow \quad 2x + 2y(x)\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$ (provided $y \neq 0$)

$\Rightarrow$ At the point $\left(\frac{3}{5}, \frac{4}{5}\right)$, $\frac{dy}{dx} = -\frac{3}{4}$ #

Remark: One cannot solve y as a function of x near the points $(1,0)$ and $(-1,0)$ which correspond to "$y=0$".

eg 2 Consider $S: x^3+z^2+ye^{xz}+z\cos y=0$

Given that z can be regarded as a function $z=z(x,y)$ of (independent) variables x, y locally near the point $(0,0,0)$. (Clearly $(0,0,0)\in S$)

Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(0,0,0)$

Soln:

$$\frac{\partial}{\partial x}\left(x^3+z^2+ye^{xz}+z\cos y\right)=0$$

$$3x^2+2z\frac{\partial z}{\partial x}+ye^{xz}\left(z+x\frac{\partial z}{\partial x}\right)+\frac{\partial z}{\partial x}\cos y=0$$

$$(3x^2+yze^{xz})+(2z+xye^{xz}+\cos y)\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \frac{\partial z}{\partial x}=-\frac{3x^2+yze^{xz}}{2z+xye^{xz}+\cos y} \quad \left(\text{provided } 2z+xye^{xz}+\cos y\neq 0\right)$$

$$\Rightarrow \frac{\partial z}{\partial x}(0,0)=0$$

Similarly $\frac{\partial}{\partial y}(x^3+z^2+ye^{xz}+z\cos y)=0$

(Check!)
$$\Rightarrow \frac{\partial z}{\partial y}=\frac{z\sin y-e^{xz}}{2z+xye^{xz}+\cos y} \quad \left(\text{provided } 2z+xye^{xz}+\cos y\neq 0\right)$$

$$\Rightarrow \frac{\partial z}{\partial y}(0,0)=-1$$

※

Finding Extrema (Maximum & Minimum)

Def: Let $\begin{cases} \bullet\ f: A \to \mathbb{R}, & A \subset \mathbb{R}^n \text{ (may not be open)} \\ \bullet\ \vec{a} \in A \end{cases}$

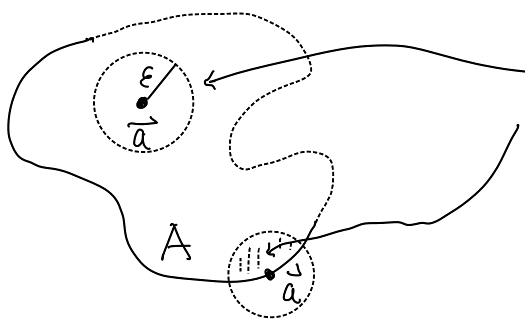
(1) f is said to have a <u>global (absolute) maximum at $\vec{a}$</u> if $f(\vec{a}) \geq f(\vec{x}) \quad \forall \vec{x} \in A$

(2) f is said to have a <u>local (relative) maximum at $\vec{a}$</u> if $f(\vec{a}) \geq f(\vec{x}) \quad \forall \vec{x} \in A$ "near" $\vec{a}$

(i.e. $\exists\, \varepsilon > 0$ s.t. $f(\vec{a}) \geq f(\vec{x}),\ \forall \vec{x} \in A \cap B_\varepsilon(\vec{a})$)

(3) Similar definitions for <u>global (absolute) minimum</u> and <u>local (relative) minimum</u> by changing the inequality to $f(\vec{a}) \leq f(\vec{x})$.

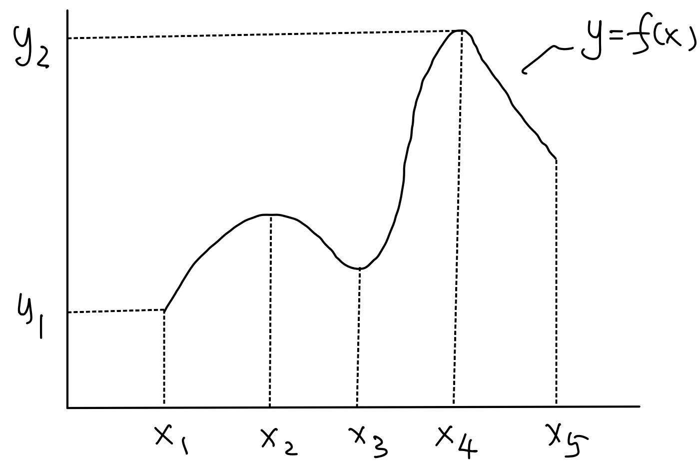
global max at $\vec{a}$ means $f(\vec{a}) \geq f(\vec{x})$ on the whole A



local max at $\vec{a}$ means $f(\vec{a}) \geq f(\vec{x})$ only holds inside the ball.

Remark: Global <u>Extremum</u> (max/min) is also a local extremum.

<u>eg1</u> $f : [x_1, x_5] \to \mathbb{R}$



Global: max : x_4; min : x_1

Local: max : x_2, x_4; min : x_1, x_3, x_5

max value : y_2

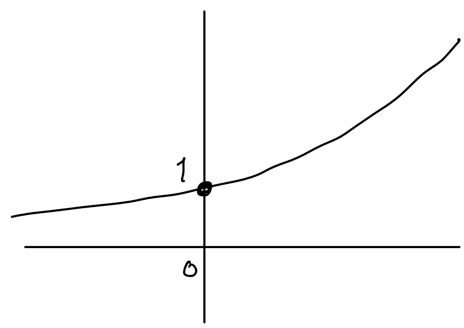
min value : y_1

(Remark : $g = f\big|_{(x_1, x_5]} : (x_1, x_5] \to \mathbb{R}$, has no global min.)

<u>eg 2</u> (<u>NOT</u> every function has global max/min)

(i) $f(x) = e^x$ on $\mathbb{R}$ (Domain <u>unbounded</u>)

No global min : $\lim_{x \to -\infty} f(x) = 0$



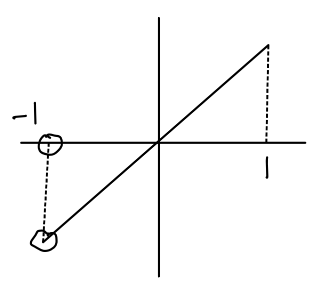
$\left(\begin{array}{l} \Rightarrow \forall a \in \mathbb{R},\ f(a) = e^a > 0 \\ \text{Then "limit"} \Rightarrow \exists x \in \mathbb{R} \text{ s.t.} \\ \quad f(x) < f(a) \end{array} \right)$

No global max : $\lim_{x \to +\infty} f(x) = +\infty$ ($\Rightarrow \forall a \in \mathbb{R}, \exists x$ s.t. $f(x) > f(a)$)

(ii) $f(x) = x$ on $(-1, 1]$ (Domain <u>not closed</u>)

Has global max at $x = 1$

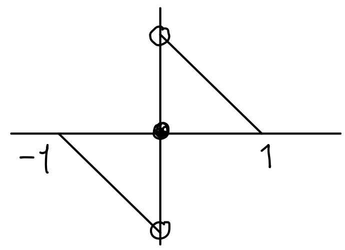
(with max. value $= f(1) = 1$)



but no global min : $\forall a \in (-1, 1]$,

$\exists\, b = \frac{a-1}{2} \in (-1, 1]$ s.t $f(b) < f(a)$.

(iii) $f = \begin{cases} 1-x, & 0<x\le 1 \\ 0, & x=0 \\ -1-x, & -1\le x<0 \end{cases}$ (discontinuous)



No global max nor global min

(Ex! similar argument as in (ii))

> <u>Extreme Value Thm</u> (EVT)
>
> Let $\begin{cases} \bullet\ A \subseteq \mathbb{R}^n \text{ be } \underline{\text{closed and bounded}}, \\ \bullet\ f: A \to \mathbb{R} \text{ be } \underline{\text{continuous}} \end{cases}$
>
> Then f has <u>global max and min</u>.

(Proof: Omitted)

<u>Remarks</u>: (1) "Compact" = closed and bounded

(2) The Thm is a sufficient, but not a necessary condition.

> <u>Def</u>: Let $\begin{cases} \bullet\ f: A \to \mathbb{R},\ A \subseteq \mathbb{R}^n \text{ (not necessary open)} \\ \bullet\ \vec{a} \in \mathrm{Int}(A) \end{cases}$
>
> Then $\vec{a}$ is called a <u>critical point</u> of f if
>
> either (1) $\vec{\nabla} f(\vec{a})$ DNE (does not exist)
>
> or (2) $\vec{\nabla} f(\vec{a}) = \vec{0}$

(i.e. either "$\frac{\partial f}{\partial x_i}(\vec{a})$ DNE for <u>some</u> $i=1,\cdots,n$" or "$\frac{\partial f}{\partial x_i}(\vec{a})=0$ for <u>all</u> $i=1,\cdots,n$")

Thm (First Derivative Test)

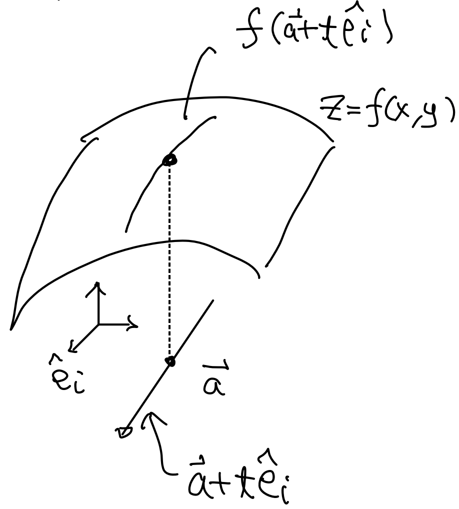
Suppose $f: A \to \mathbb{R}$ ($A \subset \mathbb{R}^n$) attains a local extremum at $\vec{a} \in \mathrm{Int}(A)$, then $\vec{a}$ is a critical point of f.

Pf: Suppose f has a local extremum at $\vec{a} \in \mathrm{Int}(A)$

If $\vec{\nabla} f(\vec{a})$ DNE, then $\vec{a}$ is a critical point.

If $\vec{\nabla} f(\vec{a})$ exists, then

$\frac{\partial f}{\partial x_i}(\vec{a})$ exists, $\forall i = 1, \cdots, n$



$\forall i$, consider the 1-variable function

$$g_i(t) = f(\vec{a} + t\hat{e}_i),$$

where $\hat{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ ith component

Then $g_i(0) = f(\vec{a}) \leq$ or $\geq f(\vec{a} + t\hat{e}_i)$ for $|t|$ small enough s.t. $\vec{a} + t\hat{e}_i$ is "near" $\vec{a}$

$\Rightarrow$ $t = 0$ is a local max/min

1-variable theory $\Rightarrow$ $0 = g_i'(0) = \left.\frac{d}{dt}\right|_{t=0} f(\vec{a} + t\hat{e}_i)$

$$= \frac{\partial f}{\partial x_i}(\vec{a}) \quad (\forall i)$$

$\Rightarrow$ $\vec{\nabla} f(\vec{a}) = \vec{0}$

$\#$

Strategy for finding Extrema

$f: A \to \mathbb{R}$

(1) Find critical points of f in $\mathrm{Int}(A)$.

(2) Study f on boundary ∂A :

find max/min of f on ∂A.

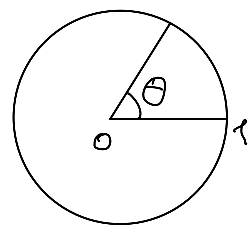
(3) Compare values of f at points found in steps (1) & (2).

---

eg 1 Find global max/min of

$$f(x,y) = \underbrace{x^2+2y^2-x+3}_{\text{cts}} \quad \text{for} \quad \underbrace{x^2+y^2 \le 1}_{\text{closed \& bdd}} \quad (A = \{x^2+y^2 \le 1\})$$

Soln : Step 1



A= closed unit disk

Critical points in $\mathrm{Int}(A)$

$$= \mathrm{Int}\{x^2+y^2 \le 1\}$$

$$= \{x^2+y^2<1\}$$

f is a polynomial $\Rightarrow$ $\vec{\nabla} f$ always exist

In fact $\vec{\nabla} f = (2x-1, 4y)$

$$\vec{0} = \vec{\nabla} f \Leftrightarrow \begin{cases} 2x-1=0 \\ 4y=0 \end{cases} \Leftrightarrow (x,y) = (\tfrac{1}{2}, 0)$$ the only critical pt in $\mathrm{Int}(A)$

$$(\|(x,y)\| = \tfrac{1}{2} < 1)$$

& $f(\frac{1}{2},0) = (\frac{1}{2})^2 + 0 - \frac{1}{2} + 3 = \frac{11}{4}$ (check!)

Step 2 Study f on $\partial\{x^2+y^2 \le 1\} = \{x^2+y^2=1\}$

Parametrize the boundary $\{x^2+y^2=1\}$ by angle θ

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases} \qquad \theta \in [0, 2\pi]$$

Therefore on $\partial\{x^2+y^2=1\}$,

$$f(\theta) = f((\cos\theta, \sin\theta)) = \cos^2\theta + 2\sin^2\theta - \cos\theta + 3$$

$$= -\cos^2\theta - \cos\theta + 5$$

$$= -(\cos\theta + \tfrac{1}{2})^2 + \frac{21}{4} \qquad \text{(check!)}$$

max. value of f on $\partial A = \frac{21}{4}$

at $\cos\theta = -\frac{1}{2} \Rightarrow (x,y) = (-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$

min. value of f on $\partial A = -(1+\frac{1}{2})^2 + \frac{21}{4} = 3$ (check!)

at $\cos\theta = 1 \Rightarrow (x,y) = (1,0)$.

Step3 Compare value of f at points from ~~steps~~ (1) & (2)

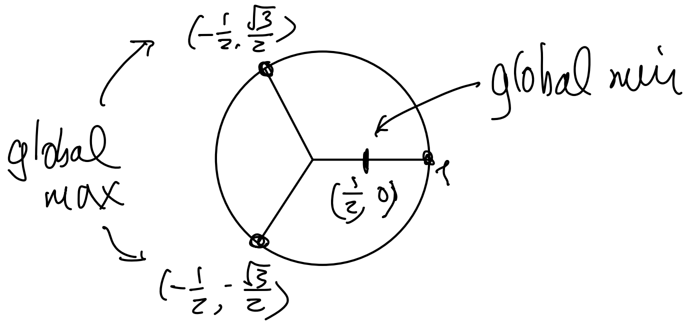
$f(\frac{1}{2}, 0) = \frac{11}{4}$ (value of critical pt in Int(A))

$f(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = \frac{21}{4}$ (max value of f on ∂A)

$f(1,0) = 3$ (min value of f on ∂A)

$\Rightarrow$ max value $= \frac{21}{4}$ at global max pts $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

min value $= \frac{11}{4}$ at the global min pt $\left(\frac{1}{2}, 0\right)$



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