Application of Chain Rule

Level Set

Recall: $f: \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$

$$\vec{x} \longrightarrow c \in \mathbb{R}, \quad c = f(\vec{x})$$

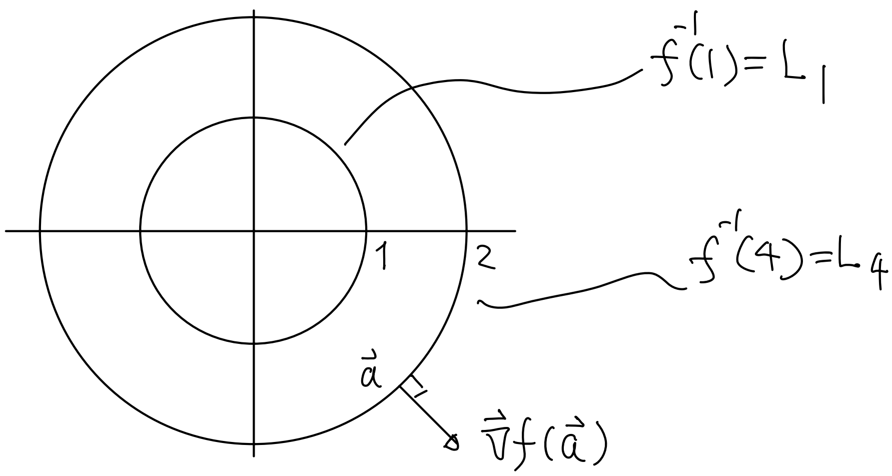
$$L_c = f^{-1}(c) = \{\vec{x} \in \Omega : f(\vec{x}) = c\}.$$

$\uparrow$ Level set of f (at level c).

eg: $f(x,y) = x^2 + y^2$

$$f^{-1}(1) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

$$f^{-1}(4) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$$



$$\left(\begin{array}{l} f^{-1}(0) = \{(0,0)\} \\ f^{-1}(-1) = \emptyset \end{array} \right)$$

Thm Let
- $f: \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$, open)
- $c \in \mathbb{R}$
- $\vec{a} \in S = f^{-1}(c)$ (S = a (nonempty) level set of f)

Suppose
- f is <u>differentiable</u> at $\vec{a}$,
- $\vec{\nabla} f(\vec{a}) \neq \vec{0}$

Then $\vec{\nabla} f(\vec{a}) \perp S$ at $\vec{a}$ (perpendicular / normal)

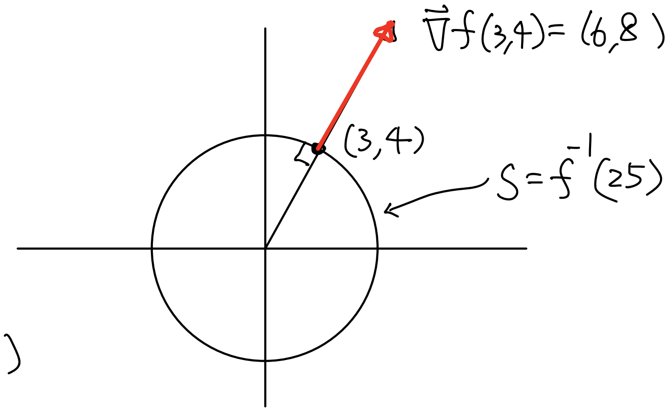
eg 1 $f(x,y) = x^2+y^2$

$\vec{\nabla} f = (2x, 2y)$

At $(3,4) \in S = f^{-1}(25)$

(i.e. level $c=25$)

$\vec{\nabla} f(3,4) = (6,8) \perp S$ at the point $(3,4)$ #



eg 2 $f(x,y) = x^2 - y^2$

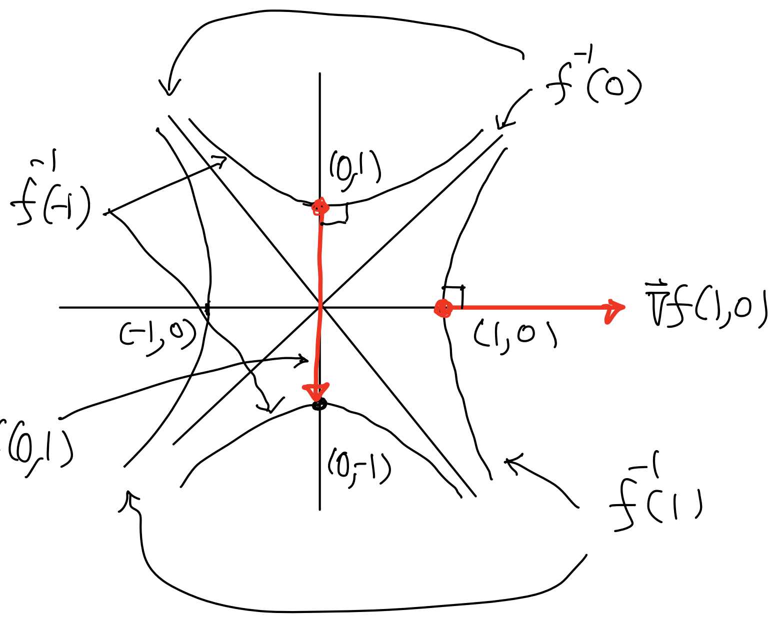
$\vec{\nabla} f(x,y) = (2x, -2y)$

$\vec{\nabla} f(1,0) = (2, 0)$

$\vec{\nabla} f(0,1) = (0, -2)$

(Ex: Try other pts)

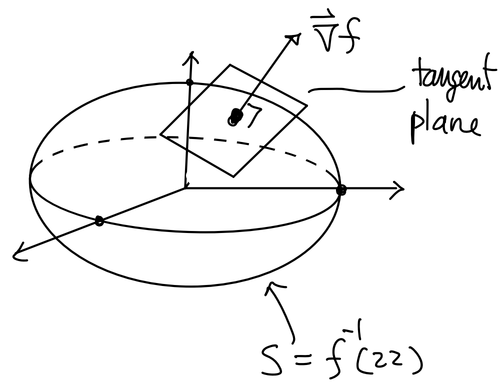
(What happen at $(0,0) \in f^{-1}(0)$)



eg 3 $S: x^2+4y^2+9z^2 = 22$ (Ellipsoid)

Find equation of tangent plane of S at the point $(3,1,1)$

(Check: $(3,1,1)$ is on S)



Soln: Let $f(x,y,z) = x^2+4y^2+9z^2$

Then $S = f^{-1}(22)$

$\vec{\nabla} f = (2x, 8y, 18z)$

$\Rightarrow \quad \vec{\nabla} f(3,1,1) = (6, 8, 18) \perp S$ at $(3,1,1)$

i.e. $\vec{\nabla} f(3,1,1)$ is a normal to the tangent plane at $(3,1,1)$

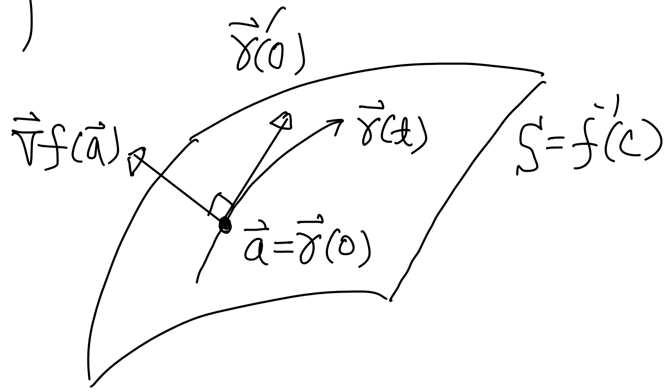
$$\Rightarrow \quad \vec{\nabla} f(3,1,1) \cdot (x-3, y-1, z-1) = 0$$

i.e. $$6(x-3) + 8(y-1) + 18(z-1) = 0$$

or $$3x + 4y + 9z = 22 \quad \text{(Check!)}$$

is the required equation of the tangent plane at $(3,1,1)$. ※

<u>Proof of the Thm:</u> $(\vec{\nabla} f \perp S = f^{-1}(c))$



Let $\vec{\gamma}(t)$ be a curve on S passing through the point $\vec{a}$ such that $\vec{\gamma}(0) = \vec{a}$

Then $f(\vec{\gamma}(t)) = c$, $\forall t$ (because $\vec{\gamma}(t) \in f^{-1}(c) = S$)

Chain rule $$0 = \left.\frac{d}{dt}\right|_{t=0} f(\vec{\gamma}(t)) = \left.\vec{\nabla} f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)\right|_{t=0}$$

$$= \vec{\nabla} f(\vec{a}) \cdot \vec{\gamma}'(0)$$

tangent vector of $\vec{\gamma}(t)$ which is also a tangent vector to S

$\vec{\nabla} f(\vec{a}) \perp$ all curves on S at $\vec{a}$

$\Rightarrow \quad \vec{\nabla} f(\vec{a}) \perp S$ at $\vec{a}$ ※