Gradient and Directional Derivative

Def: Let
- $f : \Omega \to \mathbb{R}$, ($\Omega \subseteq \mathbb{R}^n$, open)
- $\vec{a} \in \Omega$

Then the gradient vector of f at $\vec{a}$ is defined to be

$$\vec{\nabla} f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \dots, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$$

Remark: Using $\vec{\nabla} f$, linearization of f at $\vec{a}$ can be written as

$$\boxed{\begin{aligned} L(\vec{x}) &= f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) \\ &= f(\vec{a}) + \vec{\nabla} f(\vec{a}) \cdot (\vec{x} - \vec{a}) \end{aligned}}$$

eg: $f(x,y) = x^2 + 2xy$

$$\frac{\partial f}{\partial x} = 2x + 2y \qquad \frac{\partial f}{\partial y} = 2x$$

$$\therefore \quad \vec{\nabla} f(x,y) = (2x+2y, 2x)$$

(eg. $\vec{\nabla} f(1,2) = (6,2)$)

Def: Let
- $f : \Omega \to \mathbb{R}$, ($\Omega \subseteq \mathbb{R}^n$, open)
- $\vec{a} \in \Omega$
- $\vec{u} \in \mathbb{R}^n$ be a unit vector, i.e. $\|\vec{u}\| = 1$.

Then the directional derivative of f in the direction of $\vec{u}$ at $\vec{a}$ is defined to be

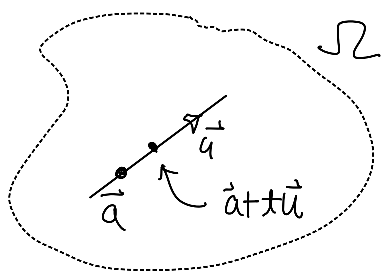
$$D_{\vec{u}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t}$$

(= rate of change of f in the direction of $\vec{u}$ at the point $\vec{a}$)

Remark: If $\vec{u} = (0, \cdots, 1, \cdots, 0) = \vec{e}_j$,

jth component

$j = 1, \cdots, n$



$$D_{\vec{e}_j} f(\vec{a}) = \frac{\partial f}{\partial x_j}(\vec{a})$$

Thm Suppose f is differentiable at $\vec{a}$.
Let $\vec{u}$ be a unit vector in $\mathbb{R}^n$, then

$$D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u}$$

eg: Let $f(x,y) = \sin^{-1}\left(\frac{x}{y}\right)$.

Find the rate of change of f at $(1,\sqrt{2})$ in the direction of $\vec{v} = (1,-1)$ (not necessary unit).

Remark: $\vec{v} \neq \vec{0} \in \mathbb{R}^n$, not necessary unit, then the direction of $\vec{v}$ is $\frac{\vec{v}}{\|\vec{v}\|}$ (a unit vector).

Soln: Let $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{1^2+(-1)^2}}(1,-1) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\sin^{-1}\left(\frac{x}{y}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{1}{y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\sin^{-1}\left(\frac{x}{y}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(-\frac{x}{y^2}\right)$$

[Note: $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous "near" $(1,\sqrt{2}) \Rightarrow f$ is C^1 "near" $(1,\sqrt{2})$]

f is differentiable at $(1,\sqrt{2})$

$$\overset{\text{Thm}}{\Rightarrow} \quad D_{\vec{u}}f(1,\sqrt{2}) = \vec{\nabla}f(1,\sqrt{2})\cdot\vec{u}$$

$$= \left(\frac{\partial f}{\partial x}(1,\sqrt{2}), \frac{\partial f}{\partial y}(1,\sqrt{2})\right)\cdot\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$$

$$= \cdots = \frac{1}{\sqrt{2}} + \frac{1}{2} \qquad \text{(Check!)}$$

Pf: (Differentiable $\Rightarrow$ $D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u}$)

Let $L(\vec{x})$ be the linearization of $f(\vec{x})$ at $\vec{a}$

then $f(\vec{x}) = L(\vec{x}) + \mathcal{E}(\vec{x})$

$$= f(\vec{a}) + \vec{\nabla} f(\vec{a}) \cdot (\vec{x} - \vec{a}) + \mathcal{E}(\vec{x})$$

with $\dfrac{|\mathcal{E}(\vec{x})|}{\|\vec{x} - \vec{a}\|} \to 0$ as $\vec{x} \to \vec{a}$.

Putting $\vec{x} = \vec{a} + t\vec{u}$, we have

$$f(\vec{a}+t\vec{u}) - f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot (t\vec{u}) + \mathcal{E}(\vec{a}+t\vec{u})$$

$$\Rightarrow \quad \frac{f(\vec{a}+t\vec{u}) - f(\vec{a})}{t} = \vec{\nabla} f(\vec{a}) \cdot \vec{u} + \frac{\mathcal{E}(\vec{a}+t\vec{u})}{t}$$

Since $\dfrac{|\mathcal{E}(\vec{a}+t\vec{u})|}{|t|} = \dfrac{|\mathcal{E}(\vec{a}+t\vec{u})|}{\|\vec{a}+t\vec{u}-\vec{a}\|}$ (because $\|\vec{u}\|=1$)

$\to 0$ as $t \to 0$ ($\Leftrightarrow \vec{a}+t\vec{u} \to \vec{a}$)

by differentiability

$$\therefore \quad D_{\vec{u}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a}+t\vec{u}) - f(\vec{a})}{t} \text{ exists}$$

and $= \vec{\nabla} f(\vec{a}) \cdot \vec{u}$

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Geometric Meanings of Gradient $\vec{\nabla} f$

> At a point $\vec{a}$, f increases (decreases) most rapidly in the direction of $\vec{\nabla} f(\vec{a})$ $(-\vec{\nabla} f(\vec{a}))$ at a rate of $\|\vec{\nabla} f(\vec{a})\|$

Idea: If f is differentiable at $\vec{a}$, then

$$D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u} \qquad (\text{for } \|\vec{u}\| = 1)$$

Cauchy-Schwarz $\Rightarrow$

$$|D_{\vec{u}} f(\vec{a})| \leq \|\vec{\nabla} f(\vec{a})\| \, \|\vec{u}\|$$

$$= \|\vec{\nabla} f(\vec{a})\|$$

i.e. $$-\|\vec{\nabla} f(\vec{a})\| \leq D_{\vec{u}} f(\vec{a}) \leq \|\vec{\nabla} f(\vec{a})\|$$

"=" holds $\Leftrightarrow \vec{u} = -\dfrac{\vec{\nabla} f(\vec{a})}{\|\vec{\nabla} f(\vec{a})\|}$

"=" holds $\Leftrightarrow \vec{u} = \dfrac{\vec{\nabla} f(\vec{a})}{\|\vec{\nabla} f(\vec{a})\|}$

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Remark: $D_{\vec{v}} f(\vec{a})$ can be defined for any vector $\vec{v}$, not necessary $\|\vec{v}\| = 1$ and could be $\vec{0}$, by the same definition

$$D_{\vec{v}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t}$$

One can show that

$$D_{\vec{v}} f(\vec{a}) = \begin{cases} \|\vec{v}\| \, D_{\frac{\vec{v}}{\|\vec{v}\|}} f(\vec{a}), & \text{if } \vec{v} \neq \vec{0} \\ 0, & \text{if } \vec{v} = \vec{0} \end{cases}$$ (if exists)

and that

$$\boxed{D_{\vec{v}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{v}}$$ if f is <u>differentiable</u> at $\vec{a}$

(not true in general, if f is not differentiable)
eg $f(x,y) = \sqrt{|xy|}$ at $(0,0)$

<u>Properties of Gradient</u>

If
- $f, g : \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$, open) are differentiable,
- c is a constant,

then

(1) $\vec{\nabla}(f \pm g) = \vec{\nabla} f \pm \vec{\nabla} g$,

(2) $\vec{\nabla}(cf) = c \vec{\nabla} f$

(3) $\vec{\nabla}(fg) = g \vec{\nabla} f + f \vec{\nabla} g$

(4) $\vec{\nabla}\left(\frac{f}{g}\right) = \frac{g \vec{\nabla} f - f \vec{\nabla} g}{g^2}$ provided $g \neq 0$

(Pf: Easily from properties of partial derivatives)

Total Differential (of real-valued function)

$f: \Omega \to \mathbb{R}$ ($\Omega \subseteq \mathbb{R}^n$, open) differentiable at $\vec{a} \in \Omega$.

Then linearization at $\vec{a}$:

$$f(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) + \varepsilon(\vec{x})$$

Usually denote: $\Delta f = f(\vec{x}) - f(\vec{a})$

$\Delta x_i = x_i - a_i$

Then $\Delta f \simeq \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a}) \Delta x_i$ $\left(\text{provided } \lim_{\vec{x} \to \vec{a}} \frac{|\varepsilon(\vec{x})|}{\|\vec{x} - \vec{a}\|} = 0\right)$

Classically, this approximation is presented as

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})\, dx_i$$

(thinking: "$\Delta f \to df$" "$\Delta x_i \to dx_i$")

Def: Let
- $f: \Omega \to \mathbb{R}$, ($\Omega \subseteq \mathbb{R}^n$, open)
- $\vec{a} \in \Omega$

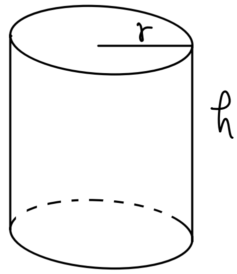
Suppose that f is differentiable on Ω. Then the total differential of f at $\vec{a}$ is defined to be the (formal) expression:

$$\boxed{df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})\, dx_i}$$

Remark: In the future, df and dx_i can be interpreted as a linear maps from $\mathbb{R}^n$ to $\mathbb{R}$.

eg: Let $V(r, h) = \pi r^2 h$

(Volume of the Cylinder)



V is differentiable

(because V is C^1)

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$= 2\pi r h \, dr + \pi r^2 dh$$

For application: Suppose we want to approximate the change of V when (r, h) changes $(r, h) = (3, 12)$ to $(3+0.08, 12-0.3)$

Then let $dr = \Delta r = 0.08$

$dh = \Delta h = -0.3$

we have

$$\Delta V \simeq dV = 2\pi r h \, dr + \pi r^2 dh$$

$$= 2\pi \cdot 3 \cdot 12 \cdot 0.08 + \pi \cdot 3^2 \cdot (-0.3)$$

$$= 3.06\pi$$

$$\approx 9.61 \quad \divideontimes$$

Properties of Total Differential

If
- $f, g : \Omega \to \mathbb{R}$ ($\Omega \subseteq \mathbb{R}^n$, open) are differentiable, and
- $c \in \mathbb{R}$ is a constant.

Then

(1) $d(f \pm g) = df \pm dg$,

(2) $d(cf) = c\,df$

(3) $d(fg) = g\,df + f\,dg$

(4) $d\left(\frac{f}{g}\right) = \frac{g\,df - f\,dg}{g^2}$ provided $g \neq 0$

(Pf: Easily from properties of partial derivatives)

Summary (on differentiation of a real-valued function on $\mathbb{R}^n$)

$f: \mathbb{R}^n \to \mathbb{R}$

A. Types of differentiations (derivatives)

- Directional Derivative :

$$D_{\vec{u}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t} \qquad (\|\vec{u}\| = 1)$$

- Partial derivatives :

$$\frac{\partial f}{\partial x_i}(\vec{a}) = D_{\vec{e}_i} f(\vec{a}), \quad \vec{e}_i = (0, \cdots, \underset{\uparrow \text{ ith component}}{1}, \cdots 0)$$

- Gradient

$$\vec{\nabla} f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \cdots, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$$

- Total Differential

$$df(\vec{a}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a}) \, dx_i$$

- Higher Derivatives

$$\frac{\partial^{k_1 + \cdots + k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}(\vec{a})$$

(provided f is C^k, $k = k_1 + \cdots + k_n$)

$\uparrow$ all partial derivatives up to order k exist & cts.

B. Linear approximation

- $L(\vec{x}) = f(\vec{a}) + \vec{\nabla} f(\vec{a}) \cdot (\vec{x} - \vec{a})$

- $f(\vec{x}) = L(\vec{x}) + \underbrace{\varepsilon(\vec{x})}_{\text{error term}}$

- f is differentiable at $\vec{a}$

$$\Longleftrightarrow \quad \lim_{\vec{x} \to \vec{a}} \frac{|\varepsilon(\vec{x})|}{\|\vec{x} - \vec{a}\|} = 0$$

In this case, $df \simeq \Delta f$ (by identifying $dx_i = \Delta x_i$)

C. Relations among various concepts

- $C^\infty \Rightarrow \cdots \Rightarrow C^{k+1} \Rightarrow C^k \Rightarrow \cdots \Rightarrow C^1 \Rightarrow C^0$ (No reverse implication)

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$$\begin{array}{ccc}
 & f \text{ is } C^1 \text{ on an open set containing } \vec{a} & \\
 & \Downarrow \; \not\Uparrow & \\
 & f \text{ is differentiable at } \vec{a} & \\
\Swarrow \; \not\nearrow & & \Searrow \; \not\nwarrow \\
D_{\vec{u}} f(\vec{a}) \text{ exists } \forall \vec{u} \in \mathbb{R}^n, \|\vec{u}\| = 1 & \begin{array}{c} \not\Rightarrow \\ \not\Leftarrow \end{array} & \begin{array}{c} f \text{ is continuous} \\ \text{at } \vec{a} \end{array} \\
\Searrow \; \not\nwarrow & & \not\Downarrow \; \not\Uparrow \\
 & \dfrac{\partial f}{\partial x_i}(\vec{a}) \text{ exists}, \ \forall i = 1, \cdots, n &
\end{array}$$

Counter examples:

eg1: $f: \mathbb{R} \to \mathbb{R}$ (in MATH2050)

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

f is differentiable on $\mathbb{R}$ but (check!)

$f'(x)$ is not continuous at $x=0$

(For $x \neq 0$, $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ limit DNE as $x \to 0$)

Similarly $g(x) = x^{2k-2} f(x)$ is k-time differentiable but

$g^{(k)}(x)$ is not continuous at $x=0$ (Pf: Omitted)

Hence k-time differentiable $\not\Rightarrow C^k$.

(For multi-variable: $h(\vec{x}) = h(x_1, \cdots, x_n) = g(x_1)$)

eg2
$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x^2+y^2 = 0 \end{cases}$$

$D_{\vec{u}} f(0,0)$ exists, $\forall$ unit vector $\vec{u} = (\cos\theta, \sin\theta) \in \mathbb{R}^2$

but f is not continuous at $(0,0)$ (check!)

eg3: $f(x,y) = |x+y|$ is continuous on $\mathbb{R}^2$, but

$f_x(0,0)$, $f_y(0,0)$ DNE (check!)

eg 4 : $f(x,y) = \sqrt{|xy|}$

$f_x(0,0)$, $f_y(0,0)$ exist (in fact $=0$)

but $D_{\vec{u}} f(0,0)$ DNE for $\vec{u} \neq \pm\vec{e}_1, \pm\vec{e}_2$

Review: Matrix Multiplication

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ be an $m \times n$-matrix

$= \begin{bmatrix} - \vec{a}_1 - \\ \vdots \\ - \vec{a}_m - \end{bmatrix}$ where $\vec{a}_i = (a_{i1}, \cdots a_{in}) \in \mathbb{R}^n$

If

$b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} | \\ \vec{b} \\ | \end{bmatrix}$ be a $n \times 1$-matrix regarded as a column vector in $\mathbb{R}^n$,

then (matrix multiplication)

$$Ab = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} - \vec{a}_1 - \\ \vdots \\ - \vec{a}_m - \end{bmatrix} \begin{bmatrix} | \\ \vec{b} \\ | \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_1 + \cdots + a_{1n}b_n \\ \vdots \\ a_{m1}b_1 + \cdots + a_{mn}b_n \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \cdot \vec{b} \\ \vdots \\ \vec{a}_m \cdot \vec{b} \end{bmatrix}$$

(result = $m \times 1$-matrix = column m-vector)

Similarly, for multiplication of $(1 \times n)$ & $(n \times k)$ matrices

$$\begin{bmatrix} - \vec{a} - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_k \\ | & & | \end{bmatrix}$$

$(\underset{\text{row vector}}{\vec{a}}, \underbrace{\vec{b}_1, \cdots, \vec{b}_k}_{\text{column vectors}} \in \mathbb{R}^n)$

$$= [\vec{a} \cdot \vec{b}_1, \cdots, \vec{a} \cdot \vec{b}_k]$$

(result = $1 \times k$-matrix = row k-vector)

In general: $(m \times n)$ times $(n \times k)$

$$AB = \begin{bmatrix} - & \vec{a}_1 & - \\ & \vdots & \\ - & \vec{a}_m & - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_k \\ | & & | \end{bmatrix} \quad \Big(\underbrace{\vec{a}_1, \cdots, \vec{a}_m}_{\text{row vectors}}, \underbrace{\vec{b}_1, \cdots, \vec{b}_k}_{\text{column vectors}} \in \mathbb{R}^n \Big)$$

$$= \begin{bmatrix} \vec{a}_1 \cdot \vec{b}_1 & \cdots & \vec{a}_1 \cdot \vec{b}_k \\ \vdots & & \vdots \\ \vec{a}_m \cdot \vec{b}_1 & \cdots & \vec{a}_m \cdot \vec{b}_k \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ A\vec{b}_1 & \cdots & A\vec{b}_k \\ | & & | \end{bmatrix} \quad \left(= A \begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_k \\ | & & | \end{bmatrix} \right)$$

$$= \begin{bmatrix} - & \vec{a}_1 B & - \\ & \vdots & \\ - & \vec{a}_m B & - \end{bmatrix} \quad \left(= \begin{bmatrix} - & \vec{a}_1 & - \\ & \vdots & \\ - & \vec{a}_m & - \end{bmatrix} B \right)$$

eg:

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}} = \begin{bmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{bmatrix} \quad \text{(check!)}$$

$$A \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 21 \\ 47 \end{bmatrix}, \quad A \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 24 \\ 54 \end{bmatrix}, \quad A \begin{bmatrix} 7 \\ 10 \end{bmatrix} = \begin{bmatrix} 27 \\ 61 \end{bmatrix}$$

$$[1, 2] B = [21, 24, 27]$$

$$[3, 4] B = [47, 54, 61]$$

Differentiability of Vector-Valued Functions

$$\vec{f} : \Omega \to \underset{\cup}{\mathbb{R}^m} \quad , (\Omega \subset \mathbb{R}^n, \text{ open })$$

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}$$

Suppose $\frac{\partial f_i}{\partial x_j}(\vec{a})$ exists for each $i = 1, \cdots, m$ & $j = 1, \cdots, n$.

$$f_i(\vec{x}) = f_i(\vec{a}) + \vec{\nabla} f_i(\vec{a}) \bullet (\vec{x} - \vec{a}) + \varepsilon_i(\vec{x}) \quad \text{——} \quad (*)_i$$

$$\Big((1\times 1) \qquad (1\times 1) \qquad \underset{\substack{\uparrow \\ \text{row}}}{(1\times n)} \cdot \underset{\substack{\uparrow \\ \text{column}}}{(n\times 1)} \qquad (1\times 1) \text{ matrix} \Big)$$

Put all $(*)_i$, we have

$$\begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix} = \underbrace{\begin{bmatrix} f_1(\vec{a}) \\ \vdots \\ f_m(\vec{a}) \end{bmatrix} + \underbrace{\begin{bmatrix} - \vec{\nabla} f_1(\vec{a}) - \\ \vdots \\ - \vec{\nabla} f_m(\vec{a}) - \end{bmatrix}}_{\substack{m\times n \text{ matrix} \\ \text{of } \left[\frac{\partial f_i}{\partial x_j}\right]_{\substack{i=1,\cdots,m \\ j=1,\cdots,n}}}} \begin{bmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{bmatrix}}_{\vec{L}(\vec{x})} + \underbrace{\begin{bmatrix} \varepsilon_1(\vec{x}) \\ \vdots \\ \varepsilon_m(\vec{x}) \end{bmatrix}}_{\text{Errors.}}$$