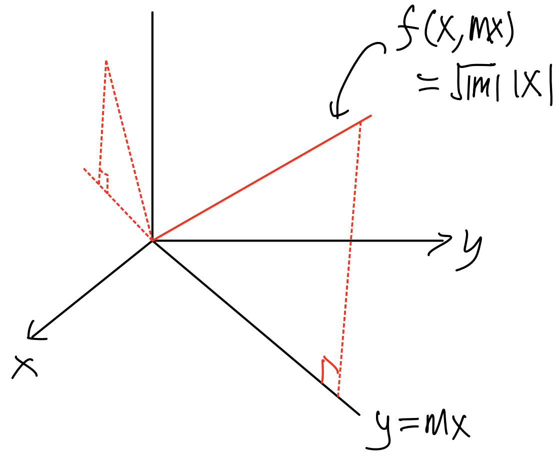
(Cont'd) Hence f is not differentiable at $(0,0)$.



<u>Remark</u>: In this example, along the straight line $y=mx$

$$f(x,y)=\sqrt{|xmx|}=\sqrt{|m|}\,|x|$$

(only "approximated" by $L(x,y)$ in the $m=0$ situation.)

(<u>Note</u>: "Differentiability" $\Rightarrow$ we can approximate all (infinity many) directions by information along x & y direction. $(x_1,\cdots,x_n)$ in general)

> <u>Thm</u> If $f(\vec{x})$ is <u>differentiable</u> at $\vec{a}$, then $f(\vec{x})$ is <u>continuous</u> at $\vec{a}$.

<u>Pf</u>: $f(\vec{x})=L(\vec{x})+\varepsilon(\vec{x})$ is differentiable $\Leftrightarrow$ $\dfrac{|\varepsilon(\vec{x})|}{\|\vec{x}-\vec{a}\|}\to 0$ as $\vec{x}\to\vec{a}$.

$$=f(\vec{a})+\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\vec{a})(x_i-a_i)+\varepsilon(\vec{x})$$

$(\vec{x}=(x_1,\cdots,x_n)$ & $\vec{a}=(a_1,\cdots,a_n))$

$$|f(\vec{x})-f(\vec{a})|=\left|\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\vec{a})(x_i-a_i)+\varepsilon(\vec{x})\right|$$

$$\leq\left|\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\vec{a})(x_i-a_i)\right|+|\varepsilon(\vec{x})| \quad \text{(triangle inequality)}$$

$$\leq\sqrt{\sum_{i=1}^{n}\left[\frac{\partial f}{\partial x_i}(\vec{a})\right]^2}\sqrt{\sum_{i=1}^{n}(x_i-a_i)^2}+|\varepsilon(\vec{x})|$$

$$= \left(\sqrt{\sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i}(\vec{a}) \right]^2} + \frac{|\varepsilon(\vec{x})|}{\|\vec{x}-\vec{a}\|} \right) \|\vec{x}-\vec{a}\|$$

$\longrightarrow 0$ as $\vec{x} \to \vec{a}$ by Squeeze's Thm & differentiability of f

$\therefore$ f is continuous at $\vec{x} = \vec{a}$

#

<u>Thm</u> If $f, g : \Omega \to \mathbb{R}$ ($\Omega \subseteq \mathbb{R}^n$, open) are <u>differentiable</u> at $\vec{a} \in \Omega$,

then (1) $f(\vec{x}) \pm g(\vec{x})$, $cf(\vec{x})$, $f(\vec{x})g(\vec{x})$ are <u>differentiable</u> at $\vec{a}$.

(2) $\dfrac{f(\vec{x})}{g(\vec{x})}$ is <u>differentiable</u> at $\vec{a}$ if $g(\vec{a}) \neq 0$

(3) (Special case of <u>Chain Rule</u>)

For 1-variable function $h(x)$ <u>differentiable</u> at $f(\vec{a})$, $h \circ f$ is <u>differentiable</u> at $\vec{a}$.

A sufficient condition for differentiability:

Thm Let $\Omega \subseteq \mathbb{R}^n$ be open, f be C^1 on Ω, then f is differentiable on Ω

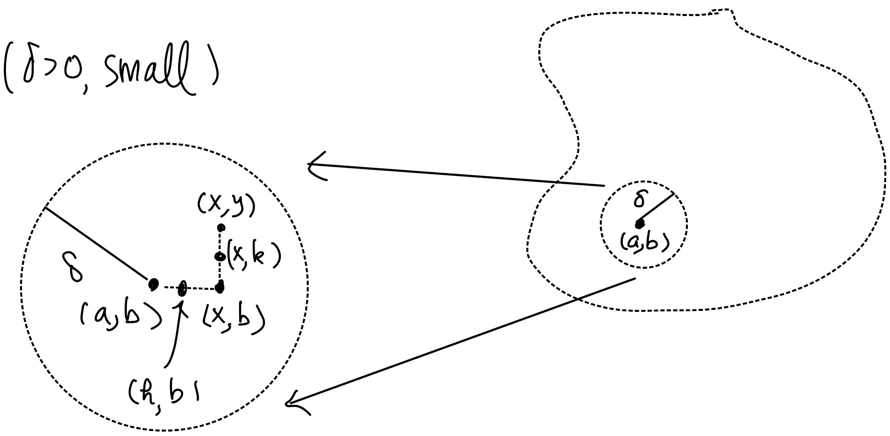
(The assumption requires all $\frac{\partial f}{\partial x_i}$ exist on Ω, not just at a single pt. $\vec{a}$)

Pf: (We prove it for 2-variables, similar proof for general case)

Suppose $(a,b) \in \Omega$

& $B_\delta(a,b) \subset \Omega$ ($\delta > 0$, small)

For any $(x,y) \in B_\delta(a,b)$



$$f(x,y) - f(a,b) = \underbrace{f(x,y) - f(x,b)} + \underbrace{f(x,b) - f(a,b)}$$

$$= f_y(x,k)(y-b) + f_x(h,b)(x-a) \quad \text{(by Mean Value Theorem)}$$

where k between y & b ; h between x & a.

$$\frac{|\varepsilon(x,y)|}{\|(x,y)-(a,b)\|} = \frac{|f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)|}{\|(x,y)-(a,b)\|}$$

$$= \frac{|f_y(x,k)(y-b) + f_x(h,b)(x-a) - f_x(a,b)(x-a) - f_y(a,b)(y-b)|}{\|(x,y)-(a,b)\|}$$

$$= \frac{\left| \left(f_x(h,b) - f_x(a,b)\right)(x-a) + \left(f_y(x,k) - f_y(a,b)\right)(y-b) \right|}{\|(x,y)-(a,b)\|}$$

$$\begin{pmatrix}\text{Cauchy-} \\ \text{Schwarz}\end{pmatrix} \leq \frac{\sqrt{\left(f_x(h,b)-f_x(a,b)\right)^2 + \left(f_y(x,k)-f_y(a,b)\right)^2}\ \sqrt{(x-a)^2+(y-b)^2}}{\|(x,y)-(a,b)\|}$$

$$= \sqrt{\left(f_x(h,b)-f_x(a,b)\right)^2 + \left(f_y(x,k)-f_y(a,b)\right)^2}$$

Note that if $(x,y) \to (a,b)$, then $(h,k) \to (a,b)$.

Hence

$$\frac{|\varepsilon(x,y)|}{\|(x,y)-(a,b)\|} \leq \sqrt{\left(f_x(h,b)-f_x(a,b)\right)^2 + \left(f_y(x,k)-f_y(a,b)\right)^2} \to 0$$

as $(x,y) \to (a,b)$

because f_x & f_y are continuous.

$\therefore$ f is differentiable at (a,b).

Since $(a,b) \in \Omega$ is arbitrary, f is differentiable on Ω #

egs: (1) constant functions $f(\vec{x}) = c$ is differentiable

(2) coordinate functions $f(\vec{x}) = x_i$ are differentiable

(3) (1)&(2) $\Rightarrow$ $f(\vec{x}) = a + b_1 x_1 + \cdots + b_n x_n$ is differentiable

(Linear function. For this f, what is the linearization $L(\vec{x})$ at $\vec{x} = \vec{0}$?)

(4) Polynomials & rational functions are differentiable (in their domain of definition).

(5) If $f(\vec{x})$ is differentiable, then $e^{f(\vec{x})}$, $\sin(f(\vec{x}))$, $\cos(f(\vec{x}))$ are differentiable.

And $\left.\begin{array}{ll} \ln(f(\vec{x})) & \text{when } f(\vec{x})>0 \\ \sqrt{f(\vec{x})} & \text{when } f(\vec{x})>0 \\ |f(\vec{x})| & \text{when } f(\vec{x})\neq 0 \\ \ln|f(\vec{x})| & \text{when } f(\vec{x})\neq 0 \end{array}\right\}$ are differentiable.

In particular, for example $\dfrac{e^{\sqrt{4+\sin(x^2+xy)}}}{\ln(1+\cos(x^2y))}$ is differentiable in the <u>domain of definition</u>

<u>eg</u>: $f(x,y,z) = xe^{x+y} - \ln(x+z) \quad (= xe^{x+y} - \log(x+z))$

Domain of $f = \{(x,y,z)\in\mathbb{R}^3 : x+z>0\}$ (is open)

$$\frac{\partial f}{\partial x} = e^{x+y} + xe^{x+y} - \frac{1}{x+z}$$

$$\frac{\partial f}{\partial y} = xe^{x+y}$$

Note that $x+z>0$ in the domain of f

$$\frac{\partial f}{\partial z} = -\frac{1}{x+z}$$

$\Rightarrow$ f is C^1 (in its domain)

$\Rightarrow$ f is differentiable (in its domain).

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