Higher Order Partial Derivatives

$n=2$, $f(x,y) \longrightarrow \frac{\partial f}{\partial x}(x,y)$, $\frac{\partial f}{\partial y}(x,y)$ 1st order derivatives

$$\longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right), \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right)$$

2nd order derivatives

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right), \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}f(x,y)\right)$$

be careful

<u>Notations</u>: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}$, $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (f_y)_y = f_{yy}$

Of course, similarly for 3rd order derivatives: eg

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right]$$

$$= f_{yyx}$$

<u>eg</u>: $f(x,y) = x\sin y + y^2 e^{2x}$

Then
$$\begin{cases} f_x = \sin y + 2y^2 e^{2x} \\ f_y = x\cos y + 2y e^{2x} \end{cases}$$

$$\begin{cases} f_{xx} = (f_x)_x = 4y^2 e^{2x} \\ f_{xy} = (f_x)_y = \cos y + 4y e^{2x} \\ f_{yx} = (f_y)_x = \cos y + 4y e^{2x} \\ f_{yy} = (f_y)_y = -x\sin y + 2e^{2x} \end{cases}$$

eg of higher order derivatives (order ≥ 3)

Previous eg : $f(x,y) = x\sin y + y^2 e^{2x}$

$$\Rightarrow \quad f_{xy} = \cos y + 4y e^{2x} = f_{yx}$$

$\Rightarrow$ (3rd order)

$$f_{xyx} = (f_{xy})_x = (\cos y + 4ye^{2x})_x = 8ye^{2x}$$
$$= (f_{yx})_x = f_{yxx}$$
$$f_{xyy} = (f_{xy})_y = (\cos y + 4ye^{2x})_y = -\sin y + 4e^{2x}$$
$$= (f_{yx})_y = f_{yxy}$$

$\vdots$

Question Is it always true that $f_{xy} = f_{yx}$?

Answer : No.

Counterexample :

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

By definition

$$f_{xy}(0,0) = (f_x)_y(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

We need to calculate $f_x(0,h)$ (for $h \neq 0$) & $f_x(0,0)$.

For $(0,h)$ $(h\neq 0)$

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right) = \frac{(x^2+y^2)(3x^2y-y^3) - xy(x^2-y^2)(2x)}{(x^2+y^2)^2}$$

$$\Rightarrow \quad f_x(0,h) = \frac{-h^5}{h^4} = -h$$

For $(0,0)$, $\quad f_x(0,0) = \lim_{k\to 0} \frac{f(k,0)-f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$

Hence $\quad f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{-h-0}{h} = -1$

Similarly, $\quad f_y(h,0) = h \quad$ (check!)

$\qquad\qquad\quad f_y(0,0) = 0 \quad$ (check!)

$$\Rightarrow \quad f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1$$

(An easy way to see this is "$f(x,y) = -f(y,x)$")

Hence $\quad f_{xy}(0,0) \neq f_{yx}(0,0)$!

<u>Question</u>: When do we have $f_{xy} = f_{yx}$?

<u>Thm</u> (<u>Clairaut's Thm</u> / <u>Mixed Derivatives Thm</u>)

Let $f: \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$, open)

If f_{xy} & f_{yx} <u>exist</u> and are <u>continuous</u> on Ω, then

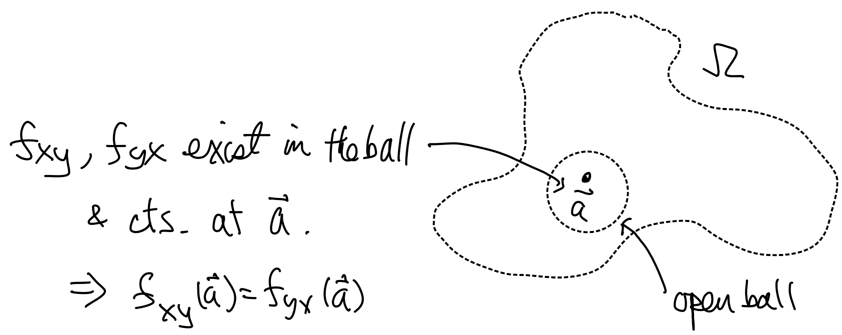
$$f_{xy} = f_{yx} \text{ on } \Omega.$$

Actually, one can prove a stronger version:

Thm Let
- $f : \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$, open)
- $\vec{a} \in \Omega$

If
- f_{xy} & f_{yx} exist in an open ball containing $\vec{a}$, and
- f_{xy} & f_{yx} are continuous at $\vec{a}$,

then $f_{xy}(\vec{a}) = f_{yx}(\vec{a})$.

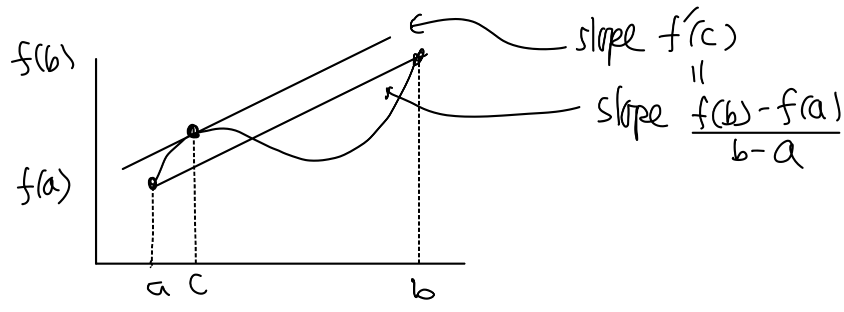


Recall

Mean Value Theorem for 1-variable Function

Let $f : [a,b] \to \mathbb{R}$,
- continuous on $[a,b]$ &
- differentiable on (a,b)

Then $\exists\, c \in (a,b)$ such that
$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

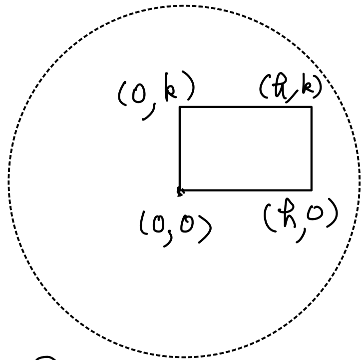


Pf of Clairaut's Thm

We may assume $\vec{a} = (0,0) \in \Omega$,

and we need to show

$$f_{xy}(0,0) = f_{yx}(0,0)$$



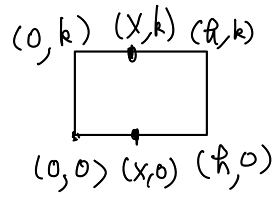
Let $h, k > 0$ and $[0,h] \times [0,k] \subset B \subset \Omega$

↑ the open ball s.t f_{xy}, f_{yx} exist (& cts at (0,0))

Define $\quad \alpha = f(h,k) - f(h,0) - f(0,k) + f(0,0)$

and $\quad g(x) = f(x,k) - f(x,0), \quad 0 \leq x \leq h$

Then $\quad \alpha = g(h) - g(0)$



$$g'(x) = f_x(x,k) - f_x(x,0)$$

Mean Value Thm $\Rightarrow \quad \exists\, h_1 \in (0,h)$ s.t.

$$\frac{g(h) - g(0)}{h} = g'(h_1) = f_x(h_1,k) - f_x(h_1,0)$$

$$\therefore \quad \alpha = h\left[f_x(h_1,k) - f_x(h_1,0) \right]$$

Mean Value Thm $\Rightarrow \quad \exists\, k_1 \in (0,k)$ s.t

$$\frac{f_x(h_1,k) - f_x(h_1,0)}{k} = (f_x)_y(h_1,k_1)$$

$\therefore \quad \alpha = hk\, f_{xy}(h_1, k_1)$

Similarly, $\exists\, (h_2, k_2) \in (0,h) \times (0,k)$ s.t.

(Ex!)

$$\alpha = hk\, f_{yx}(h_2, k_2)$$

$$\Rightarrow \quad f_{xy}(h_1, k_1) = f_{yx}(h_2, k_2)$$

Letting $h, k \to 0^+ \Rightarrow h_1, k_1 \to 0^+$ & $h_2, k_2 \to 0^+$

By continuity of f_{xy} & f_{yx} at $\vec{a} = (0,0)$, we have

$$f_{xy}(0,0) = f_{yx}(0,0) \quad \#$$

Def Let $f: \Omega \to \mathbb{R}$ ($\Omega \subseteq \mathbb{R}^n$, open)

Then
- f is called a <u>C^k function</u> if <u>all partial derivatives</u> of f up to <u>order k</u> <u>exist</u> and are <u>continuous</u> on Ω
- f is called a <u>C^∞ function</u> if f is C^k for all $k \geq 0$.

<u>egs</u>: (1) If f is continuous (0-order partial derivative) then f is C^0.

(2) If f is C^2, then $f, f_x, f_y, f_{xx}, f_{xy} = f_{yx}, f_{yy}$ exist & are all continuous. (by Clairaut's)

(3) Polynomials, Rational functions, exponential, logarithm, trigonometric functions are C^∞ function on their domains of definition & hence their

sum / difference / product / quotient / compositions

are C^∞ function on their domains of definition.

explicit eg: $e^{x^2-y}\sin\left(\frac{x}{y}\right)$ is C^∞ on domain of definition $= \mathbb{R}^2 \setminus \{x\text{-axis}\}$
(except $y=0$)

<u>Generalization of Clairaut's Thm</u>

> If f is C^k on an open set $\Omega \subseteq \mathbb{R}^n$, then the order of (taking) differentiation <u>does not matter</u> for all partial derivatives up to order k.

<u>eg</u> If $f(x,y,z)$ is C^3, then

$$f_{xz} = f_{zx}, \quad f_{xyz} = f_{xzy} = f_{zxy} = f_{zyx}$$

$\vdots$ etc.

$$= f_{yzx} = f_{yxz}$$

$f_{xxy} = f_{xyx} = f_{yxx}$ and etc.

Differentiability

Recall: 1-variable: f is differentiable at a

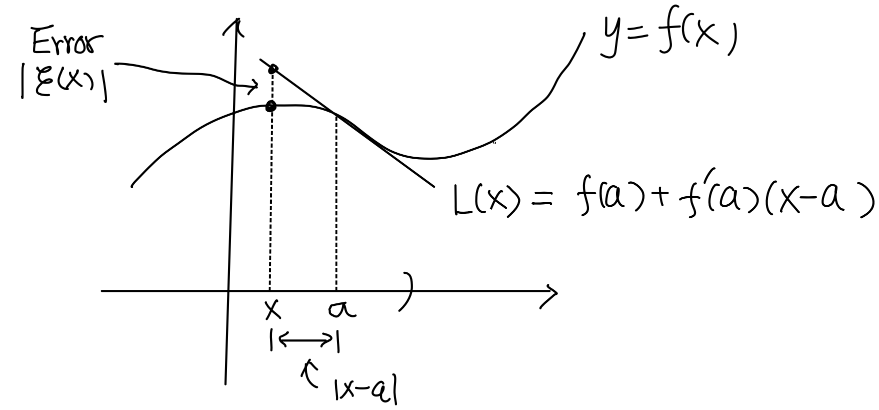
if $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists

which is equivalent to

Linear Approximation of f at the point a:

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}$$

$L(x)$ is the "best" linear function (deg ≤ 1, poly) to approximate $f(x)$ near a



What does it mean by the "best"?

Answer: $\lim_{x \to a} \frac{|f(x) - L(x)|}{|x-a|} = 0$

where $f(x) - L(x)$ is usually referred as the

$$\boxed{\text{"error" term} \quad \varepsilon(x) = f(x) - L(x).}$$

$$\left(\begin{array}{c} \lim_{x \to 0} \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| \\ \parallel \\ \lim_{x \to a} \frac{|\varepsilon(x)|}{|x-a|} = 0 \end{array} \right)$$

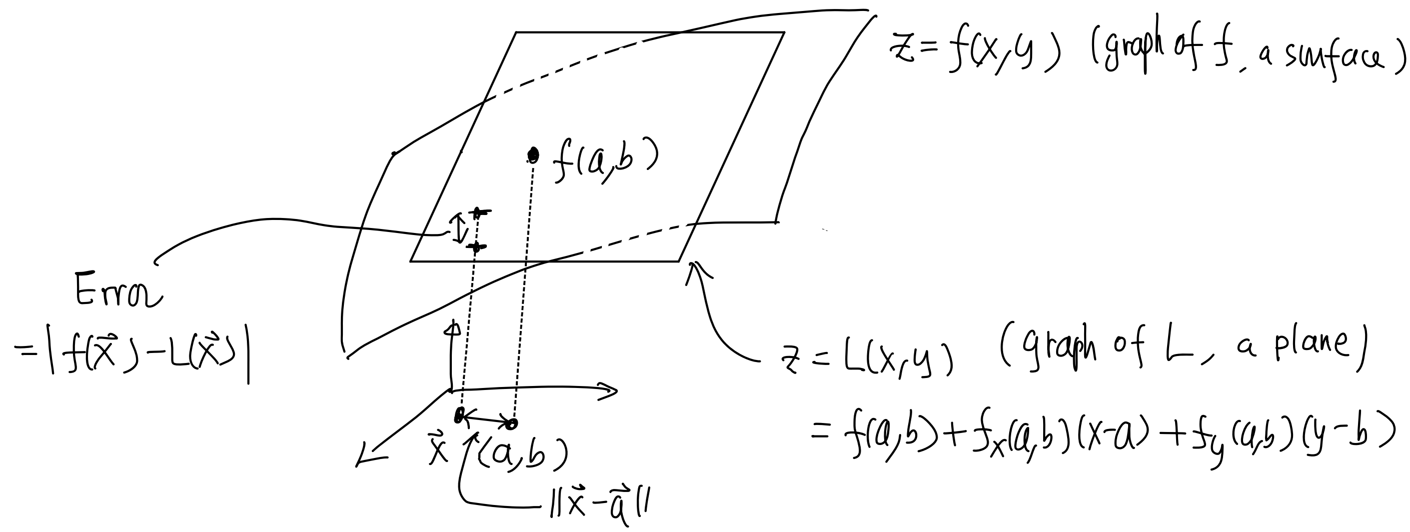
Higher dimension's analog:

linear function (deg ≤ 1, poly)

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

and want

$$f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$



Def: Let
- $f : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$, open
- $\vec{a} = (a_1, \cdots, a_n) \in \Omega$

Then f is said to be differentiable at $\vec{a}$

if (1) $\frac{\partial f}{\partial x_i}(\vec{a})$ exists for all $i = 1, \cdots, n$

(2) In the linear approximation for $f(\vec{x})$ at $\vec{a}$

$$f(\vec{x}) = \underbrace{f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i)}_{L(\vec{x}) \text{ linear approx.}} + \underset{\text{error term}}{\varepsilon(\vec{x})},$$

the error term $\varepsilon(\vec{x})$ satisfies

$$\lim_{\vec{x} \to \vec{a}} \frac{|\varepsilon(\vec{x})|}{\|\vec{x} - \vec{a}\|} = 0.$$

(A differentiable function is one which can be well approximated by a linear function locally.)

<u>Remark</u>: $L(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \underbrace{\frac{\partial f}{\partial x_i}(\vec{a})}_{\substack{\uparrow \\ \text{slope of } f \text{ in} \\ x_i\text{-direction at } \vec{a}}} \underbrace{(x_i - a_i)}_{\Delta x_i}$

- $L(\vec{x})$ is a deg ≤ 1 polynomial
- $L(\vec{a}) = f(\vec{a})$
- $\frac{\partial L}{\partial x_i}(\vec{a}) = \frac{\partial f}{\partial x_i}(\vec{a})$ (Easy Ex!)
- The graph of $y = L(\vec{x})$ is a n-plane <u>tangent</u> to the graph of $y = f(\vec{x})$ (which is a surface) at the point $\vec{x} = \vec{a}$.

<u>eg 1</u>: $f(x,y) = x^2 y$

(1) Show that f is differentiable at $(1,2)$

(2) Approximate $f(1.1, 1.9)$ using linearization,

(3) Find tangent plane of $z = f(x,y)$ at $(1, 2, \underset{f(1,2)}{2})$.

<u>Soln</u>: (1) $\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial f}{\partial y} = x^2$

$\Rightarrow \frac{\partial f}{\partial x}(1,2) = 4$, $\frac{\partial f}{\partial y}(1,2) = 1$

The linearization at $(1,2)$ is

$$L(x,y) = f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2)$$

$$= 2 + 4(x-1) + (y-2) \quad (\text{or } = 4x+y-4)$$

with error term

$$\mathcal{E}(x,y) = f(x,y) - L(x,y)$$

$$= x^2y - [2+4(x-1)+(y-2)]$$

$$\lim_{(x,y)\to(1,2)} \frac{|\mathcal{E}(x,y)|}{\|(x,y)-(1,2)\|} = \lim_{(x,y)\to(1,2)} \frac{|x^2y - [2+4(x-1)+(y-2)]|}{\sqrt{(x-1)^2+(y-2)^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{|(1+h)^2(2+k) - (2+4h+k)|}{\sqrt{h^2+k^2}} \quad \left(\text{letting } \begin{array}{l} h = x-1 \\ k = y-2 \end{array}\right)$$

$$= \lim_{(h,k)\to(0,0)} \frac{|h^2k + 2hk + 2h^2|}{\sqrt{h^2+k^2}} \quad \left(\text{let: } \begin{array}{l} h = r\cos\theta \\ k = r\sin\theta \end{array}\right)$$

$$= \lim_{r\to 0} r\left(r\cos^2\theta\sin\theta + 2\cos\theta\sin\theta + 2\cos^2\theta\right) = 0 \text{ (by Squeeze Thm)}$$

$\therefore$ f is differentiable at $(1,2)$.

(b) $((1.1, 1.9) \sim (1,2))$

$$f(1.1, 1.9) \simeq L(1.1, 1.9)$$

$$= 2 + 4(1.1-1) + (1.9-2)$$

$$= 2 + 0.4 - 0.1 = 2.3$$

(c) The equation of the tangent plane of $z = f(x,y)$ at the point $(x,y) = (1,2)$ is

$$z = L(x,y) = 2 + 4(x-1) + (y-2) \quad \#$$

$\left(\text{i.e.} \quad z = 4x + y - 4 \quad \text{or} \quad 4x + y - z = 4 \right)$

eg 2 Is $f(x,y) = \sqrt{|xy|}$ differentiable at $(0,0)$?

Soln:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

Linearization $L(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0)$

$$= 0 + 0\cdot(x-0) + 0\cdot(y-0)$$

$$\equiv 0$$

Error term $\varepsilon(x,y) = f(x,y) - L(x,y) = f(x,y) = \sqrt{|xy|}$

$$\lim_{(x,y)\to(0,0)} \frac{|\varepsilon(x,y)|}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \sqrt{|\cos\theta \sin\theta|}$$

Different directions (ie different θ) give different limits.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{|\varepsilon(x,y)|}{\|(x,y)-(0,0)\|} \quad \text{DNE}$$

(to be cont'd)