Polar Coordinates in $\mathbb{R}^2$

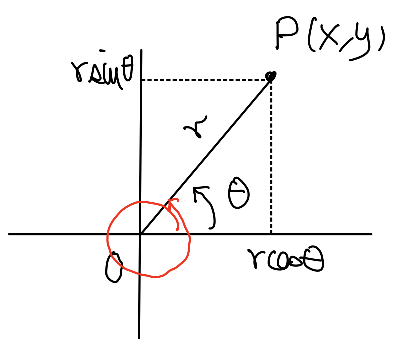
$P = (x,y) \in \mathbb{R}^2$ can be represented by

$(r\cos\theta, r\sin\theta)$

where $r = \sqrt{x^2+y^2}$ = distance from origin

θ = angle from the positive x-axis to $\overrightarrow{OP}$

in counter-clockwise direction



Remarks: (i) $(r\cos\theta, r\sin\theta) = (r\cos(\theta+2k\pi), r\sin(\theta+2k\pi))$

for any $k \in \mathbb{Z} = \{\cdots,-2,-1,0,1,2,\cdots\}$

(ii) For $P=(0,0)$, then $\begin{cases} r=0 \\ \theta \text{ is not well-defined} \end{cases}$

(iii) For our defn, we <u>usually</u> set

- $r \in [0,\infty)$ $(r \geq 0)$
- $\theta \in [0, 2\pi)$ $(0 \leq \theta < 2\pi)$

But in some book, $\begin{cases} r \in \mathbb{R} & \text{(can be negative as in Textbook)} \\ & \text{(see later examples)} \\ \theta \in \mathbb{R} \end{cases}$

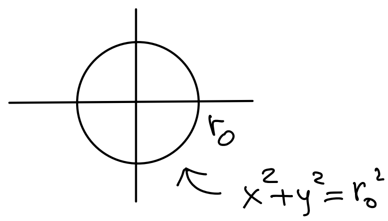
Change of Coordinates formula

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases} \quad (\text{for } x>0)$$

Curves in Polar Coordinates

eg : Circle of radius $r_0 > 0$, centered at origin

Polar Equation :



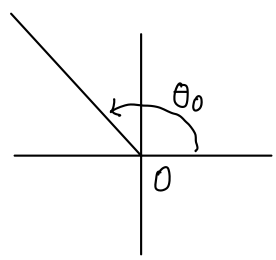
$$r = r_0 .$$

Parametric Form (in polar)

$$\begin{cases} r = r_0 \\ \theta = t , \quad t \in [0, 2\pi] \end{cases}$$

$$\left(\vec{x}(t) = (r_0\cos t, r_0 \sin t) \right)$$

eg Half ray from origin



Polar Equation

$$\theta = \theta_0$$

Parametric form (in polar)

$$\begin{cases} r = t , \quad t \in [0, \infty) \\ \theta = \theta_0 \end{cases}$$

eg: **Archimedes Spiral**

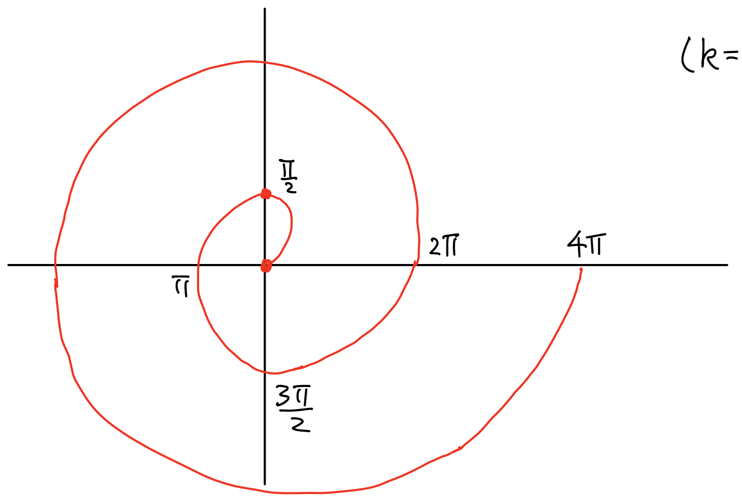
$k > 0$ is a constant.

Polar Equation

$r = k\theta$

Parametric form (in pole)

$$\begin{cases} r = kt \\ \theta = t \end{cases} \qquad t \in [0, \infty)$$

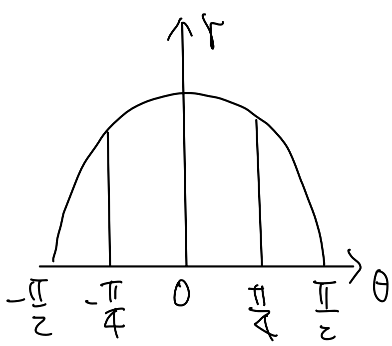


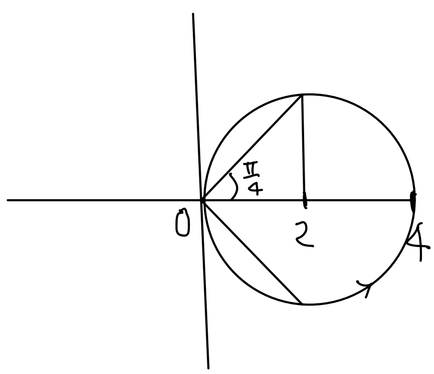
Remark: $\vec{x}(t) = (r\cos\theta, r\sin\theta) = (kt\cos t, kt\sin t) = k(t\cos t, t\sin t)$

$\Rightarrow \vec{x}'(t) = k(\cos t - t\sin t, \sin t + t\cos t)$ is the tangent vector at $\vec{x}(t) = k(t\cos t, t\sin t)$.

$\left((r'(t), \theta'(t)) = (k, 1) \text{ is } \underline{\text{not}} \text{ the tangent vector in } \mathbb{R}^2 \right)$

eg: $r = 4\cos\theta \quad (\geq 0,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$





$r = 4\cos\theta$

$\Rightarrow r^2 = 4r\cos\theta$

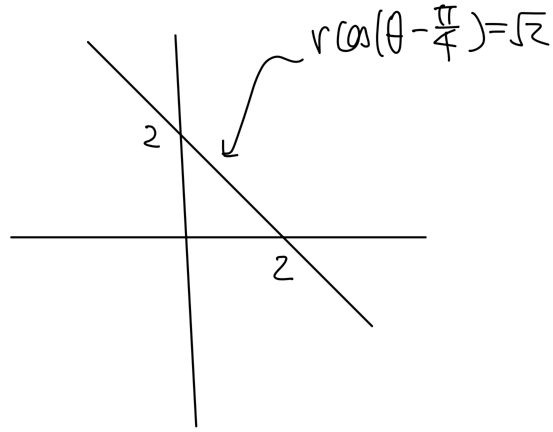
i.e. $x^2 + y^2 = 4x$

eg: $r\cos(\theta - \frac{\pi}{4}) = \sqrt{2}$

$\Leftrightarrow r(\cos\theta\cos\frac{\pi}{4} + \sin\theta\sin\frac{\pi}{4}) = \sqrt{2}$

$\Leftrightarrow \frac{r\cos\theta}{\sqrt{2}} + \frac{r\sin\theta}{\sqrt{2}} = \sqrt{2}$

$\Leftrightarrow x + y = 2$



Negative r

Our convention is $r \geq 0$.

But sometimes is convenient to allow $r < 0$ by the interpretation

$$(x, y) = (r\cos\theta, r\sin\theta)$$
$$= (-|r|\cos\theta, -|r|\sin\theta)$$
$$= -(|r|\cos\theta, |r|\sin\theta) \qquad \left(= (|r|\cos(\theta+\pi), |r|\sin(\theta+\pi))\right)$$

eg: $r = -2$, $\theta = \frac{\pi}{6}$ $\quad (x, y) = (-2\cos\frac{\pi}{6}, -2\sin\frac{\pi}{6}) = -(\sqrt{3}, 1) = (-\sqrt{3}, -1)$

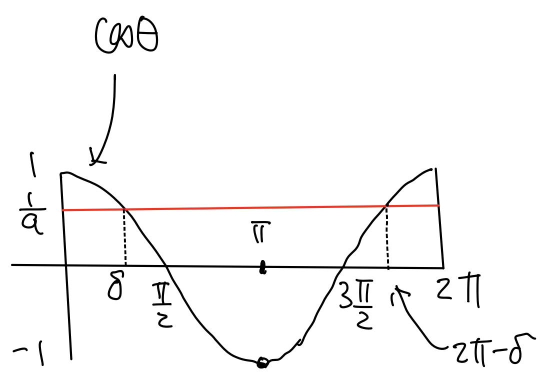
eg: $r = 1 - (1+\varepsilon)\cos\theta$, $\quad \varepsilon > 0$

$= 1 - a\cos\theta$, $\quad a = 1 + \varepsilon > 1$

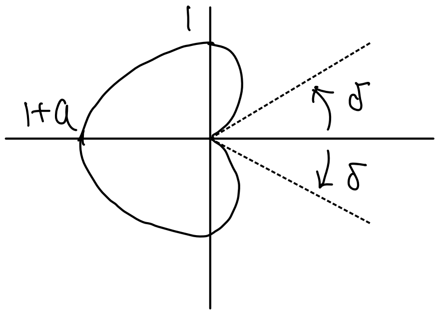
Soln:

Case 1 $\quad r \geq 0$

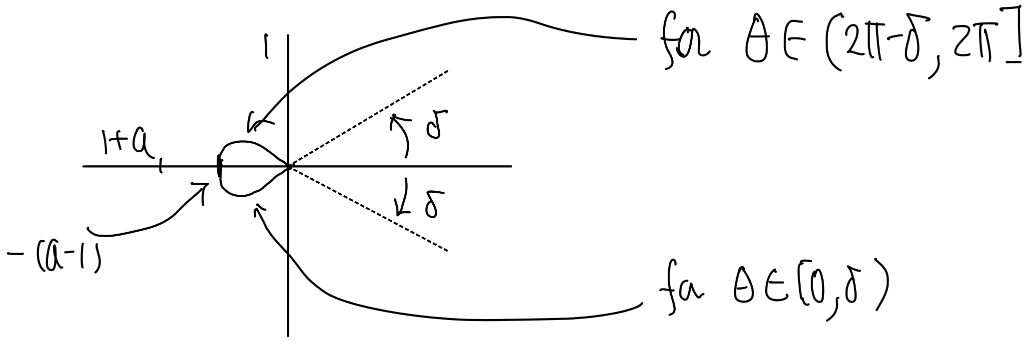
$\Rightarrow 1 - a\cos\theta \geq 0 \Rightarrow \cos\theta \leq \frac{1}{a} < 1$



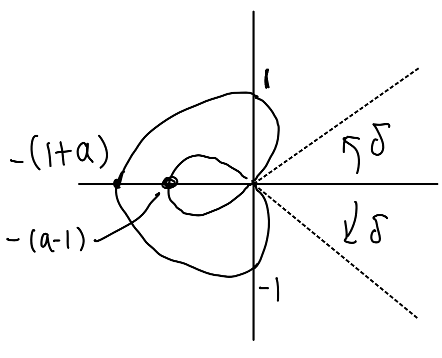
Let $\delta = \cos^{-1}\left(\frac{1}{a}\right)$, then θ can only run through the subinterval $[\delta, 2\pi - \delta]$ (of $[0, 2\pi]$)



<u>Case 2</u> $\quad r < 0 \Rightarrow 0 \leq \theta < \delta$ & $2\pi - \delta < \theta \leq 2\pi$



So if we allow $r \in \mathbb{R}$, then $r = 1 - a\cos\theta$ can be defined fa all $\theta \in [0, 2\pi]$ & the curve becomes a curve with <u>self-intersection</u>:



$(a = 1 + \varepsilon > 1)$

Coordinates Systems in $\mathbb{R}^3$ (generalizing polar coordinates)

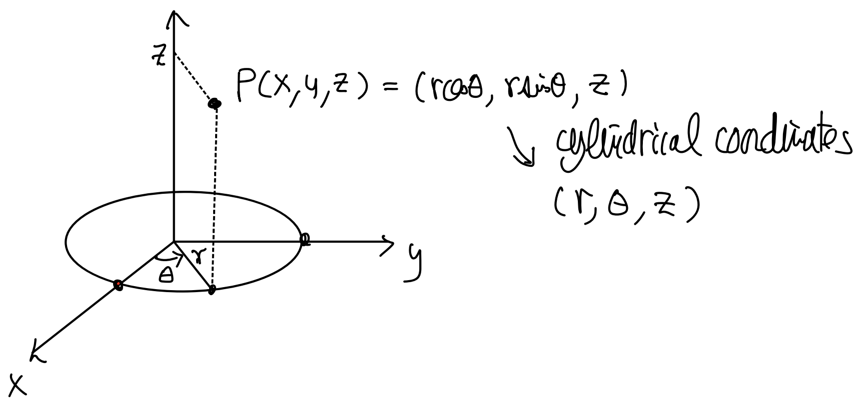
Cylindrical Coordinates

$$(x, y, z) \longrightarrow (r, \theta, z)$$

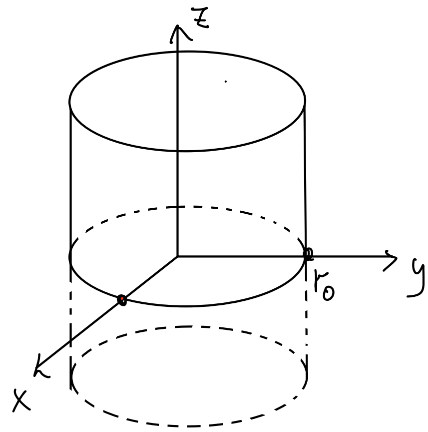
where $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ i.e. (r, θ) is polar coordinates for xy-plane.

Formula

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



eg: $r = r_0$ constant is the equation for the cylinder in the figure.



eg Helix $(a\cos t, a\sin t, bt)$ $(a > 0,\ b \in \mathbb{R})$

can be represented in cylindrical coordinates by

$$\begin{cases} r = a \\ \theta = t \\ z = bt \end{cases} \qquad t \in [0, 2\pi]$$

(Ex: Sketch the curve)

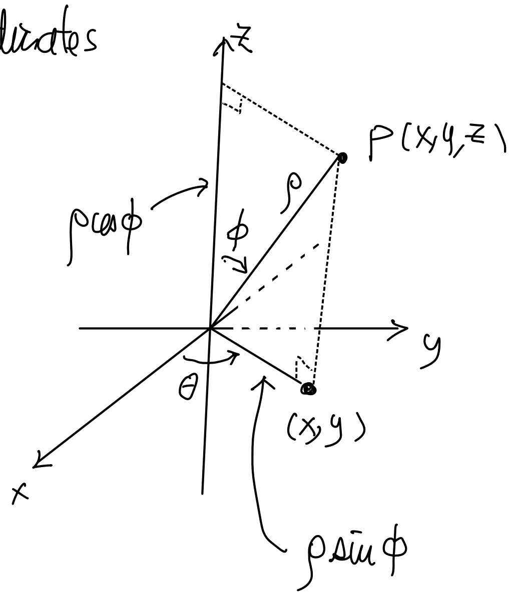
Spherical Coordinates

$P = (x, y, z) \in \mathbb{R}^3$ can be represented by

$\rho =$ distance from origin $= \sqrt{x^2+y^2+z^2}$

$\theta = \theta$ as in cylindrical coordinates

$\phi =$ angle from positive z-axis to $\overrightarrow{OP}$.



Remark: $\phi \in [0, \pi]$

Formulae $\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$

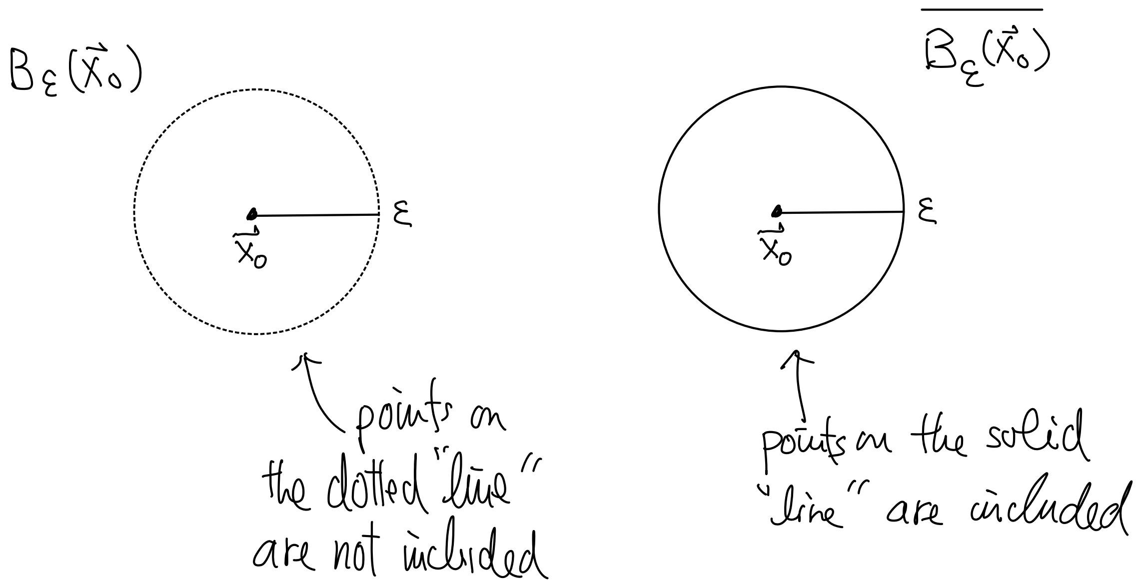
(Tutorial for Egs)

Topological Terminology in $\mathbb{R}^n$

Def • $B_\varepsilon(\vec{x_0}) = \{ \vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\| < \varepsilon \}$ is called the <u>open ball</u> of radius ε and centered at $\vec{x}_0$

• $\overline{B_\varepsilon(\vec{x_0})} = \{ \vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\| \leq \varepsilon \}$ is called the <u>closed ball</u> of radius ε and centered at $\vec{x}_0$

<u>Remark</u> : If $n=2$, $B_\varepsilon(\vec{x}_0)$, $\overline{B_\varepsilon(\vec{x}_0)}$ are referred as <u>open disk</u>, <u>closed disk</u> and denoted by $D_\varepsilon(\vec{x}_0)$, $\overline{D_\varepsilon(\vec{x}_0)}$ in some textbooks.



<u>Recall notation</u> : " $\exists$ " : there exist(s)

" $\forall$ " : for all

Def: Let S be a set in $\mathbb{R}^n$.

(1) The <u>interior of S</u> is the set

$$\mathrm{Int}(S) = \{\vec{x} \in \mathbb{R}^n : \exists\, \varepsilon > 0 \text{ s.t. } B_\varepsilon(\vec{x}) \subset S\}$$

Points in $\mathrm{Int}(S)$ are called <u>interior points</u> of S

(2) The <u>exterior of S</u> is the set

$$\mathrm{Ext}(S) = \{\vec{x} \in \mathbb{R}^n : \exists\, \varepsilon > 0 \text{ s.t. } B_\varepsilon(\vec{x}) \subset \mathbb{R}^n \setminus S\}$$

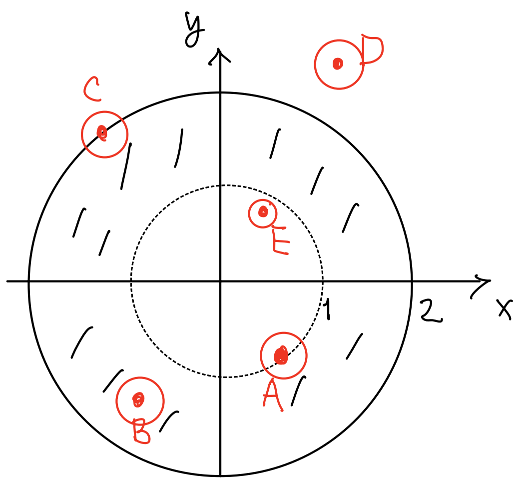
Points in $\mathrm{Ext}(S)$ are called <u>exterior points</u> of S

(3) The <u>boundary of S</u> is the set

$$\partial S = \left\{\vec{x} \in \mathbb{R}^n : \forall\, \varepsilon > 0 \text{ s.t. } \begin{array}{l} B_\varepsilon(\vec{x}) \cap S \neq \phi \ \& \\ B_\varepsilon(\vec{x}) \cap (\mathbb{R}^n \setminus S) \neq \phi \end{array} \right\}$$

Points on ∂S are called <u>boundary points</u> of S

<u>eg</u>: $S = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 4\} \subset \mathbb{R}^2$



A = boundary point

B = interior point

C = boundary point

D = exterior point

E = exterior point

$$\mathrm{Int}(S) = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$$

$$\mathrm{Ext}(S) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \ \& \ 4 < x^2 + y^2\}$$

$$\partial S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \ \& \ x^2 + y^2 = 4\}$$

Prop Let $S \subset \mathbb{R}^n$. Then

(1) $\mathbb{R}^n$ = disjoint union of $\text{Int}(S)$, $\text{Ext}(S)$ and ∂S

(2) $\text{Int}(S) \subseteq S$

$\text{Ext}(S) \subseteq \mathbb{R}^n \setminus S$

(3) A point on ∂S may or may not be in S

(For statement (3), see points A & C in the above eg.)

Def A set $S \subset \mathbb{R}^n$ is called

(1) <u>open</u> if $\forall \vec{x} \in S$, $\exists\, \varepsilon > 0$ such that $B_\varepsilon(\vec{x}) \subset S$

(2) <u>closed</u> if $\mathbb{R}^n \setminus S$ is open

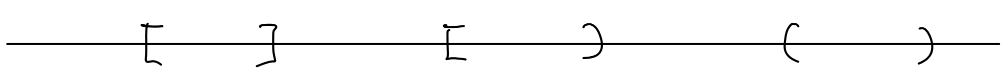
<u>Equivalent definition</u>:

(1) S open $\Leftrightarrow$ $S = \text{Int}(S)$

(2) S closed $\Leftrightarrow$ $S = \text{Int}(S) \cup \partial S$ (check!)

<u>eg</u> Is $S = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 4\}$ open or closed?

<u>Answer</u>: Not open and not closed.

(Similar to  $\mathbb{R}^1$)