eg (revisit) $\vec{a}=(2,3,5)$, $\vec{b}=(1,2,3)$. Find $\vec{a}\times\vec{b}$.

Soln: $\vec{a}=2\hat{i}+3\hat{j}+5\hat{k}$ & $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$

$$\begin{aligned}\vec{a}\times\vec{b} &= (2\hat{i}+3\hat{j}+5\hat{k})\times(\hat{i}+2\hat{j}+3\hat{k})\\ &= 2\cdot\vec{0}+3(-\hat{k})+5\hat{j}+4\hat{k}+6\cdot\vec{0}+10(-\hat{i})\\ &\quad +6(-\hat{j})+9\hat{i}+15\cdot\vec{0}\\ &= -\hat{i}-\hat{j}+\hat{k}\end{aligned}$$

$\#$

Triple Product (only in $\mathbb{R}^3$)

Let $\vec{a},\vec{b},\vec{c}\in\mathbb{R}^3$

The triple product of $\vec{a}$, $\vec{b}$ & $\vec{c}$ (order is important) is defined by $(\vec{a}\times\vec{b})\cdot\vec{c}$

Note: Assume $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)$ & $\vec{c}=(c_1,c_2,c_3)$

$$\begin{aligned}(\vec{a}\times\vec{b})\cdot\vec{c} &= \left(\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}, -\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}, \begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\right)\cdot(c_1,c_2,c_3)\\ &= c_1\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}-c_2\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}+c_3\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\end{aligned}$$

$$\therefore \boxed{(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}}$$

(expansion formula along 3rd row)

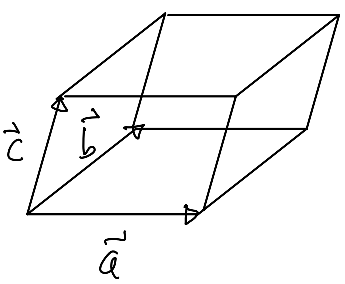
Remark: It is easy to obtain

$$(\vec{a}\times\vec{b})\cdot\vec{c}=(\vec{b}\times\vec{c})\cdot\vec{a}=(\vec{c}\times\vec{a})\cdot\vec{b} \qquad \text{(Ex)}$$

$$=-(\vec{b}\times\vec{a})\cdot\vec{c}=-(\vec{a}\times\vec{c})\cdot\vec{b}=-(\vec{c}\times\vec{b})\cdot\vec{a}$$

<u>Geometric meaning</u>

$\left| (\vec{a} \times \vec{b}) \cdot \vec{c} \right|$ = Volume of the parallelopiped spanned by $\vec{a}, \vec{b}$ & $\vec{c}$.



<u>Pf</u>:

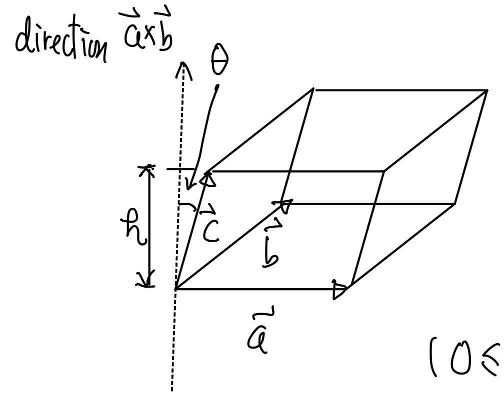
$h = \|\vec{c}\| \cos\theta$

and $(\vec{a} \times \vec{b}) \cdot \vec{c}$

$= \|\vec{a} \times \vec{b}\| \, \|\vec{c}\| \cos\theta$

$=$ Area ($\vec{b}$ ▱ $\vec{a}$) h

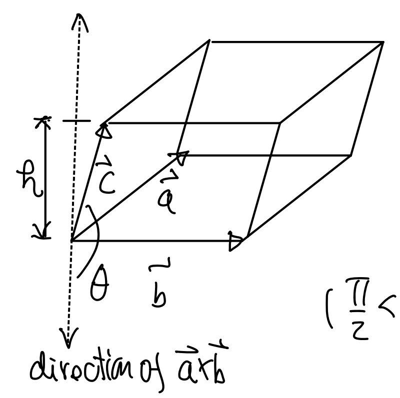
$=$ Volume of the parallelopiped



$(0 \leq \theta < \frac{\pi}{2})$

Ex: For case $\frac{\pi}{2} < \theta \leq \pi$

the proof is similar.

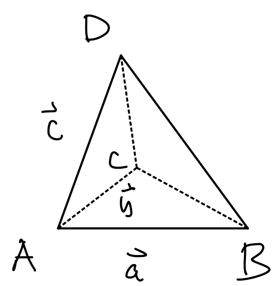


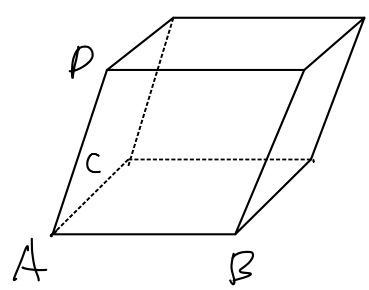
$(\frac{\pi}{2} < \theta \leq \pi)$

<u>Remarks</u>: (i) $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0 \Leftrightarrow$ Vol (parallelepiped) $= 0$

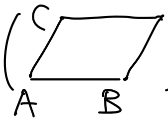
$\Leftrightarrow \{ \vec{a}, \vec{b}, \vec{c} \}$ are linearly dependent.

(ii) Tetrahedron





Vol (Tetrahedron)

$= \frac{1}{3}\,\text{Area}(\triangle ABC)\cdot\text{height} = \frac{1}{3}\cdot\frac{1}{2}\,\text{Area}\Big($  $\Big)\cdot\text{height}$

$$= \frac{1}{6}\,\text{Vol (Parallelepiped)}$$

$$= \frac{1}{6}\left|(\vec{a}\times\vec{b})\cdot\vec{c}\right|$$

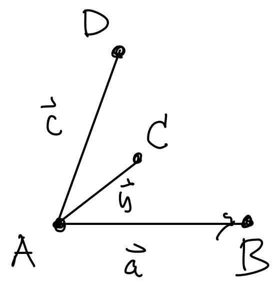
eg Let $A=(1,0,1)$, $B=(1,1,2)$, $C=(2,1,1)$, $D=(2,1,3)$

Find volume of tetrahedron $ABCD$

Soln: $\vec{a} = \overrightarrow{AB} = (1,1,2)-(1,0,1)$

$(\overrightarrow{OB} - \overrightarrow{OA})$

$= (0,1,1)$



$\vec{b} = \overrightarrow{AC} = (2,1,1)-(1,0,1)$

$= (1,1,0)$

$\vec{c} = \overrightarrow{AD} = (2,1,3)-(1,0,1) = (1,1,2)$

$$\text{Vol (Tetrahedron)} = \frac{1}{6}\left|(\vec{a}\times\vec{b})\cdot\vec{c}\right|$$

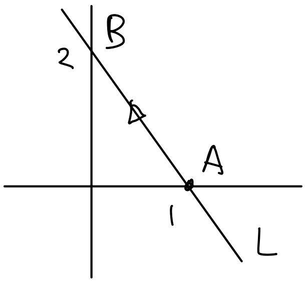
$$= \frac{1}{6}\left|\det\begin{pmatrix}0&1&1\\1&1&0\\1&1&2\end{pmatrix}\right| = \frac{1}{6}\left|-2\right| = \frac{1}{3}$$

Linear Objects in $\mathbb{R}^n$

(lines, plane, k-plane, hyperplane)

Line

eg in $\mathbb{R}^2$



Equation form: $2x + y = 2$

Parametric form:

$$(x,y) = \overrightarrow{OA} + t\,\overrightarrow{AB}, \quad t \in \mathbb{R}$$

$$= (1,0) + t\big((0,2) - (1,0)\big)$$

$$= (1-t,\ 2t)$$

i.e. $\begin{cases} x = 1 - t \\ y = 2t \end{cases}, \quad t \in \mathbb{R}$

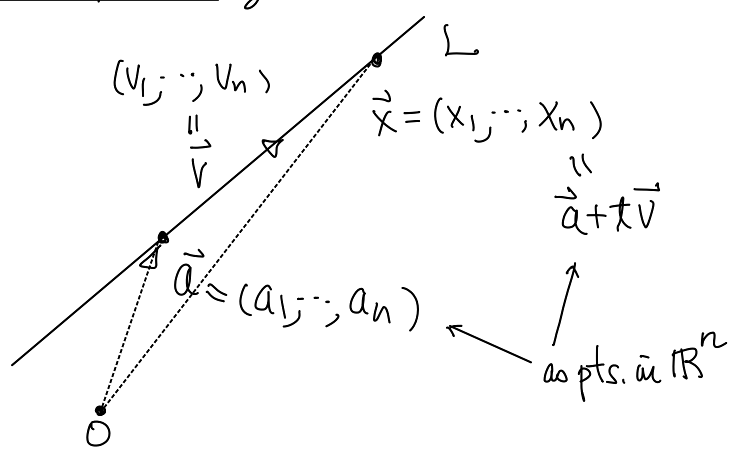
Note: Symmetric / Slope form $\quad \dfrac{x-1}{-1} = \dfrac{y-0}{2}$ (Ex!)

Parametric Form of a line in $\mathbb{R}^n$ ($n=3$ particularly)

Let L = a line in $\mathbb{R}^n$

$\vec{a}$ = a point on L

$\vec{v}$ = a direction vector of L $(\vec{v} \neq \vec{0})$



Then

> **parametric form of L**
>
> $\vec{x} = \vec{a} + t\,\vec{v}$
>
> $t \in \mathbb{R}$ called a parameter

(L is parametrized by $t \in \mathbb{R}$)

i.e. $(x_1, \cdots, x_n) = (a_1, \cdots, a_n) + t(v_1, \cdots, v_n)$

$= (a_1 + tv_1, \cdots, a_n + tv_n)$

i.e. $\begin{cases} x_1 = a_1 + tv_1 \\ \vdots \\ x_n = a_n + tv_n \end{cases} \quad t \in \mathbb{R}$

eg A line L in $\mathbb{R}^3$ passes through

$A = (1,2,3)$, $B = (-1,3,5)$

Soln: $\left(\begin{array}{lll} \text{choose} & \vec{a} = A \; (= \overrightarrow{OA}) & (\text{or } B) \text{ as vector} \\ & \vec{v} = \overrightarrow{AB} & (\text{or } \overrightarrow{BA}) \end{array} \right)$

A parametrization of L is

$$\vec{x} = (1,2,3) + t((-1,3,5) - (1,2,3))$$

$$= (1,2,3) + t(-2,1,2) \quad \divideontimes$$

(In high school notations: $x = 1-2t$, $y = 2+t$, $z = 3+2t$.)

Remarks: (i) Parametric form is not unique (many choice as in eg)

(ii) From the answer, we get symmetric form:

$$\frac{x-1}{-2} = \frac{y-2}{1} = \frac{z-3}{2} \quad (= t)$$

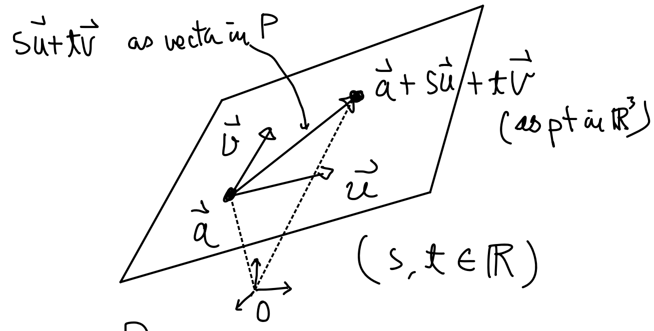
$$\Leftrightarrow \begin{cases} x-1 = -2(y-2) \\ 2(y-2) = z-3 \end{cases}$$

Planes in $\mathbb{R}^3$

(1) $P =$ a plane in $\mathbb{R}^3$

$\vec{a} =$ a point on P

$\vec{u}, \vec{v} =$ 2 linearly independent vectors on P.



$(s, t \in \mathbb{R})$

Then

Parametric Form of P

$$\vec{x} = \vec{a} + s\vec{u} + t\vec{v}$$

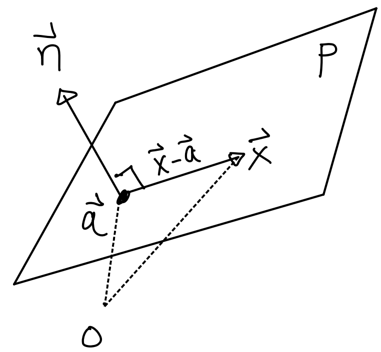
two parameters

(2) $P =$ a plane in $\mathbb{R}^3$

$\vec{a} =$ a point on P

$\vec{n} =$ a normal vector of P (orthogonal)

(i.e. $\vec{n}$ is perpendicular to P & $\vec{n} \neq \vec{0}$)



Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{n} = (n_1, n_2, n_3)$ and $\vec{x} = (x, y, z)$

$\vec{x} \in P \iff (\vec{x} - \vec{a}) \perp \vec{n}$

$\iff (\vec{x} - \vec{a}) \cdot \vec{n} = 0 \quad (\iff \vec{x} \cdot \vec{n} = \vec{a} \cdot \vec{n})$

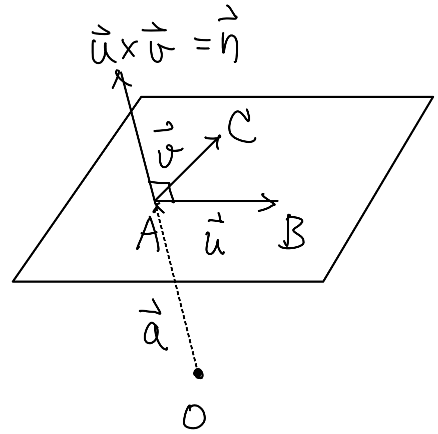
$\iff (x - a_1, y - a_2, z - a_3) \cdot (n_1, n_2, n_3) = 0$

$\iff n_1 x + n_2 y + n_3 z = \underbrace{n_1 a_1 + n_2 a_2 + n_3 a_3}_{\text{constant.}}$

$\boxed{\begin{array}{c}\underline{\text{Equation of } P} \text{ (general in } \mathbb{R}^3\text{)} \\ n_1 x + n_2 y + n_3 z = c\end{array}}$ $\quad (c = \vec{a} \cdot \vec{n})$ provided $\vec{n} = (n_1, n_2, n_3) \neq \vec{0}$.

<u>eg</u>: Suppose P is a plane (in $\mathbb{R}^3$) passing through
$A = (0,0,1)$, $B = (0,2,0)$, $C = (-1,1,0)$
Represent P using (i) parametric form; (ii) equation.

<u>Soln</u>: (i) $\left(\text{Pick } \underbrace{\vec{a} = A}, \quad B \text{ or } C \atop \vec{u}, \vec{v} = \vec{AB}, \vec{AC} \; ; \; \ldots \text{etc} \right)$



$\vec{a} = (0,0,1)$

$\vec{u} = (0,2,0) - (0,0,1) = (0,2,-1)$

$\vec{v} = (-1,1,0) - (0,0,1) = (-1,1,-1)$

Then the parametric form of P is $\vec{x} = (0,0,1) + s(0,2,-1) + t(-1,1,-1)$,
$s, t \in \mathbb{R}$.

(ii) Take $\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -1 \\ -1 & 1 & -1 \end{vmatrix} = (-1,1,2)$ (check!)

$\Rightarrow$ Eqt. of P $\quad \big((x,y,z) - (0,0,1)\big) \cdot (-1,1,2) = 0$

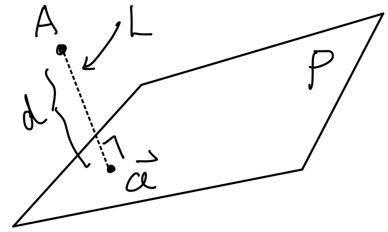
i.e. $\quad -x + y + 2z = 2 \quad$ (Check!)

(or $x - y - 2z = -2$) $\quad$ ##

eg: Find the <u>distance</u> between

$A = (2,1,1)$ and the

$P: -x + 2y - z = -4$ — (*)



<u>Soln</u>: From (*), $\vec{n} = (-1, 2, -1) \perp P$

Consider the line L (passing thro. A in the direction of $\vec{n}$)

$$\vec{x} = \vec{A} + t\vec{n}$$

$$= (2,1,1) + t(-1,2,-1)$$

$$= (2-t, 1+2t, 1-t) \qquad t \in \mathbb{R}$$

Let $\vec{a}$ be the intersection of L and P, then

$\vec{a}$ can be found as follows:

put $(x,y,z) = (2-t, 1+2t, 1-t)$ into eqt. (*) of P.

$$-(2-t) + 2(1+2t) - (1-t) = -4$$

$$\Rightarrow \quad t = -\frac{1}{2} \quad \text{(Check!)}$$

$$\Rightarrow \quad \vec{a} = \left(2-\left(-\tfrac{1}{2}\right), 1+2\left(-\tfrac{1}{2}\right), 1-\left(-\tfrac{1}{2}\right)\right)$$

$$= \left(\frac{5}{2}, 0, \frac{3}{2}\right) \qquad \text{(check!)}$$

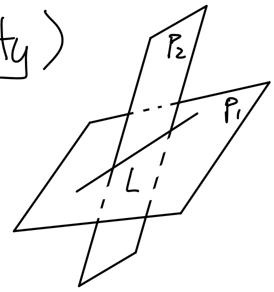
Hence distance between A & P = distance between A & $\vec{a}$

$$= \sqrt{\left(2-\frac{5}{2}\right)^2 + (1-0)^2 + \left(1-\frac{3}{2}\right)^2}$$

$$= \frac{\sqrt{6}}{2} \qquad \text{※}$$

(Of course, one can develop the distance formula using this method (p715 Text)) eg: <u>Line in $\mathbb{R}^3$ by equations</u>

Two planes intersect at a line (if not empty)



For instance $\begin{cases} x+y+6z=6 \\ x-y-2z=-2 \end{cases}$ $\left(\begin{array}{c} (1,1,6) \& (1,-1,-2) \\ \text{are linearly indep.} \end{array} \right)$
(Ex!)

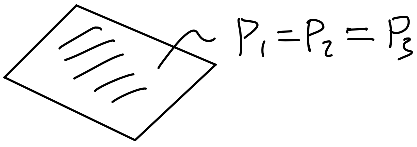
is a line. Then Gaussian Elimination will give us a parametric form of the line. i.e solving the system of linear equation by setting a variable to be a parameter: eg

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}$$ (by setting $z=t$) (linear algebra!) (Ex!)

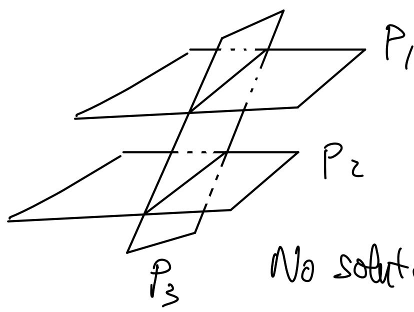
eg: How about 3 linear equations?

Linear Alg $\Rightarrow$
- Case 1: unique solution, i.e. intersection = {point}.
- Case 2: Infinitely many solutions; could be a line or a plane
- Case 3: No solution, i.e. no intersection.

eg.



infinitely many solutions



No solution

(Ex: Try other situations)

<u>Remark</u> : In n dim., a (hyper)plane is given by $\underline{\vec{x} \cdot \vec{n} = c}$.

as in planes in $\mathbb{R}^3$ ($\dim(\text{hyperplane}) = n-1$)

Then linear algebra $\Rightarrow$ all possible situations for intersections of (hyper)planes.

(Discussion omitted)