Math 2010A Advanced Calculus I

Differential Calculus of Functions of Several variables.

$$\underbrace{f(x,y)\,,\quad f(x,y,z)}_{\text{mainly}}\,,\quad \underline{f(x_1,\cdots,x_n)}$$

$n=2,3$ (most of the times)

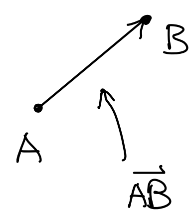
<u>Vectors in $\mathbb{R}^n$</u>: $\vec{x}=(x_1,\cdots,x_n)\in\mathbb{R}^n$

$$\mathbb{R}^n=\mathbb{R}\times\cdots\times\mathbb{R} \quad (n \text{ copies of } \mathbb{R})$$

$$=\{(x_1,\cdots,x_n):\ x_i\in\mathbb{R} \text{ for } i=1,\cdots,n\}$$

<u>Remarks</u>: (i) In Textbook, "bold face" $\boldsymbol{x}$ is used to denote a vector.

(ii) In High school, $\vec{AB}$ is a vector with initial point A & terminal point B.

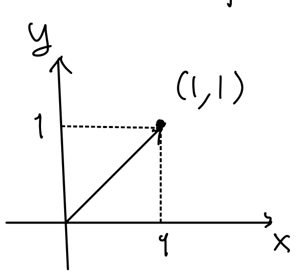


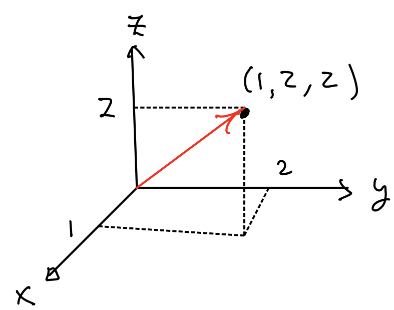
For $\vec{x}=(x_1,\cdots,x_n)$ is a vector with initial point $A=0=(0,\cdots,0)$ & terminal point $B=(x_1,\cdots,x_n)$.

(iii) $\mathbb{R}^n$ is called <u>Cartesian product</u> of n-copies of $\mathbb{R}$.

$(x_1,\cdots,x_n)$ <u>Cartesian / rectangular coordinates</u> of the point.

<u>eg</u>:





Basic Operations of Vectors

Let $\vec{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$

$\vec{b} = (b_1, \cdots, b_n) \in \mathbb{R}^n$

$r \in \mathbb{R}$

Equality

$$\vec{a} = \vec{b} \iff a_i = b_i \text{ for } i = 1, \cdots, n.$$

Addition

$$\vec{a} + \vec{b} = (a_1 + b_1, \cdots, a_n + b_n)$$

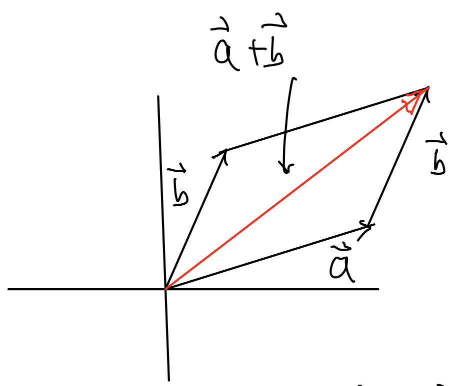
Scalar Multiplication

$$r\vec{a} = (ra_1, \cdots, ra_n)$$

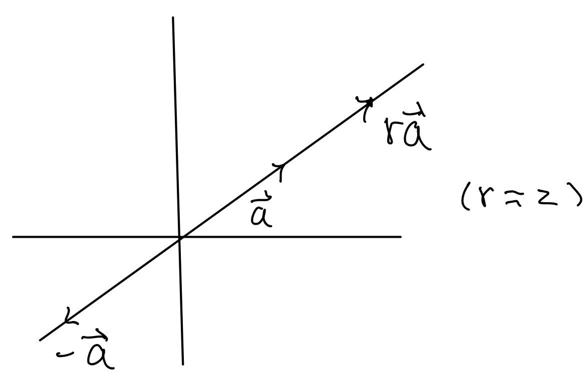
Subtraction

$$\vec{a} - \vec{b} = \vec{a} + (-1)\vec{b}$$

$$= (a_1 - b_1, \cdots, a_n - b_n)$$



(parallelogram law)



(Ex: What is $\vec{a} - \vec{b}$?)

Length, Dot (Inner) Product in $\mathbb{R}^n$

__Defn__: Let $\vec{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$
$\vec{b} = (b_1, \cdots, b_n) \in \mathbb{R}^n$

Then __dot product__ (or __inner product__)

$$\vec{a} \bullet \vec{b} = a_1 b_1 + \cdots + a_n b_n$$

(Have to write the dot!)

In particular,

$$\vec{a} \cdot \vec{a} = a_1^2 + \cdots + a_n^2 \overset{\text{def}}{=} \|\vec{a}\|^2$$

$\|\vec{a}\| = \sqrt{a_1^2 + \cdots + a_n^2}$ is called the __length__/__magnitude__ of the vector $\vec{a}$.

(In Textbook, it's denoted by $|\vec{a}|$.)

__Remark__: Another notation for dot (inner) product

$$\langle \vec{a}, \vec{b} \rangle = \vec{a} \bullet \vec{b}$$

__Properties of Dot Product__

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$, $r \in \mathbb{R}$. Then

(1) $(\vec{a} + \vec{b}) \bullet \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

$\vec{a} \bullet (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

(2) $(r\vec{a}) \bullet \vec{b} = \vec{a} \bullet (r\vec{b}) = r(\vec{a} \bullet \vec{b})$

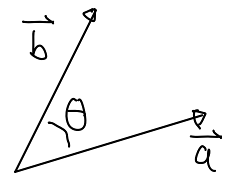
(3) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

(4) $\vec{a}\cdot\vec{a} \geq 0$ and "equality holds" $\Leftrightarrow \vec{a} = \vec{0} = (0,\cdots,0)$ zero vector

(5) $\vec{a}\cdot\vec{a} = \|\vec{a}\|^2$

(6) $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$

where θ = angle between $\vec{a}$ & $\vec{b}$



Hence, if $\vec{a},\vec{b} \neq 0$,

$$\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \vec{a} \perp \vec{b} \text{ (perpendicular)}$$

<u>Remarks</u> (i) For $n=2,3$, (5) & (6) are geometric properties of $\mathbb{R}^n$.

(ii) For $n \geq 4$, $\|\vec{a}\|$ in (5) & (6) is defined before and the θ in (6) is the definition of

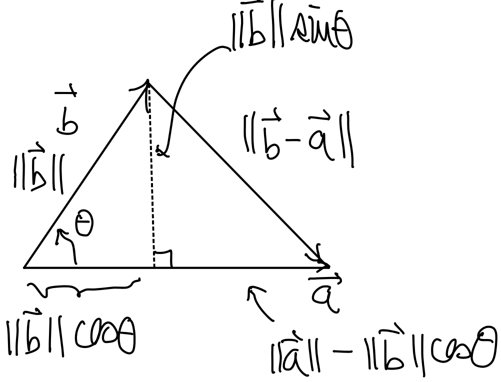
$$\theta \overset{\text{def}}{=} \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right) \text{ for high dimensional vectors.}$$

(iii) If $\|\vec{u}\| = 1$, then $\vec{u}$ is called a <u>unit vector</u>.

<u>eg</u>:
$$\left.\begin{aligned} \hat{i} &= (1,0,0) \\ \hat{j} &= (0,1,0) \\ \hat{k} &= (0,0,1) \end{aligned}\right\} \text{ are unit vectors in } \mathbb{R}^3$$

<u>Pf of (6)</u> (for $n \leq 3$)

(Assume $0 < \theta \leq \frac{\pi}{2}$)



$$\|\vec{b}-\vec{a}\|^2 = (\|\vec{a}\|-\|\vec{b}\|\cos\theta)^2 + \|\vec{b}\|^2\sin^2\theta \qquad \text{—— (1)}$$

On the other hand

$$\|\vec{b}-\vec{a}\|^2 = (\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a})$$

$$= \vec{b}\cdot\vec{b} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{a}\cdot\vec{a}$$

$$= \|\vec{b}\|^2 - 2\vec{a}\cdot\vec{b} + \|\vec{a}\|^2 \qquad \text{—— (2)}$$

$$(1)\ \&\ (2) \Rightarrow \quad \vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta \qquad \#$$

<u>eg</u>: Suppose $\vec{v}$ & $\vec{w}$ are vectors of the <u>same length</u>.

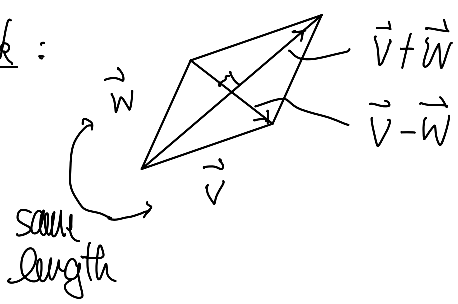
Show that $(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) = 0$.

<u>Soln</u>: $(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) = \vec{v}\cdot\vec{v} + \cancel{\vec{w}\cdot\vec{v}} - \cancel{\vec{v}\cdot\vec{w}} - \vec{w}\cdot\vec{w}$

$$= \|\vec{v}\|^2 - \|\vec{w}\|^2 = 0$$

since $\vec{v}, \vec{w}$ have same length. #

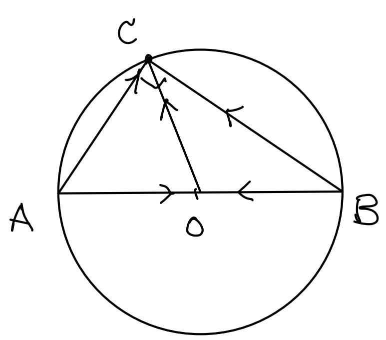
<u>Remark</u>:



Geometric meaning:

Diagonals of a rhombus are perpendicular.

<u>eg</u>:



AB is diameter, C on the circle

show that $\angle ACB = 90^\circ = \frac{\pi}{2}$

Soln :

$$\vec{AC} = \vec{AO} + \vec{OC}$$
$$\vec{BC} = \vec{BO} + \vec{OC}$$

$AB = \text{diameter} \Rightarrow \vec{AO} = -\vec{BO}$

Then
$$\begin{aligned}\vec{AC}\cdot\vec{BC} &= (\vec{AO}+\vec{OC})\cdot(\vec{BO}+\vec{OC}) \\ &= (\vec{AO}+\vec{OC})\cdot(-\vec{AO}+\vec{OC}) \\ &= -\|\vec{AO}\|^2 - \cancel{\vec{OC}\cdot\vec{AO}} + \cancel{\vec{AO}\cdot\vec{OC}} + \|\vec{OC}\|^2 \\ &= 0\end{aligned}$$

since A, C are on the circle with center O

$\Rightarrow \angle ACB = \frac{\pi}{2}$ ✱

Two inequalities (Au's Book §1.2)

Cauchy-Schwarz Inequality

Let $\vec{a}, \vec{b} \in \mathbb{R}^n$. Then $|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|\|\vec{b}\|$

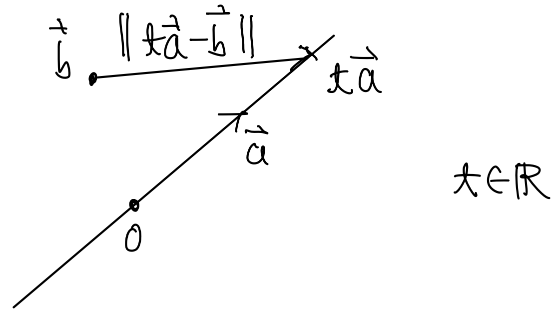
i.e. $\left|\sum_{i=1}^n a_i b_i\right| \leq \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2}$

Equality holds $|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|\|\vec{b}\|$ $\Longleftrightarrow$ $\vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$ for some $r \in \mathbb{R}$

Pf: Case 1 If $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then both sides are 0, equality holds & $\vec{a}=0\vec{b}$ or $\vec{b}=0\vec{a}$

Case 2 $\vec{a}\neq\vec{0}$ & $\vec{b}\neq\vec{0}$

Let $f(t)=\|t\vec{a}-\vec{b}\|^2\geq 0$ for $t\in\mathbb{R}$



Then $0\leq f(t)=(t\vec{a}-\vec{b})\cdot(t\vec{a}-\vec{b})=\cdots$

$$=t^2\|\vec{a}\|^2-2t\,\vec{a}\cdot\vec{b}+\|\vec{b}\|^2$$

$$\Rightarrow \quad \Delta=(-2\vec{a}\cdot\vec{b})^2-4\|\vec{a}\|^2\|\vec{b}\|^2\leq 0$$

(discriminant of the quadratic)

$$\Rightarrow \quad |\vec{a}\cdot\vec{b}|\leq\|\vec{a}\|\|\vec{b}\|.$$

Finally, "Equality" holds $\Leftrightarrow \Delta=0$

$\Leftrightarrow f(t)$ has repeated root $r\in\mathbb{R}$

$\Leftrightarrow \|r\vec{a}-\vec{b}\|^2=f(r)=0$

$\Leftrightarrow \vec{b}=r\vec{a}$ ※

Remark: Cauchy-Schwarz inequality $\Rightarrow$

$$-1\leq\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\leq 1 \qquad (\text{provided } \vec{a}\neq\vec{0} \text{ \& } \vec{b}\neq\vec{0})$$

$\Rightarrow$ The formula defining the angle between $\vec{a}$ & $\vec{b}$ in high-dim.

$$\theta=\cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right) \text{ is well-defined.}$$