13.8 **5. Constrained minimum** Find the points on the curve $xy^2 = 54$ nearest the origin.

Sol'n: $f(x,y) = x^2+y^2$. $g(x,y) = xy^2$

Then using Lagrange Multipliers, consider $F(x,y,\lambda) = f(x,y) - \lambda(g(x,y)-54)$

$$= x^2+y^2-\lambda(xy^2-54).$$

$$\vec{\nabla}F(x,y,\lambda)=0 \Leftrightarrow \begin{cases} 0 = \frac{\partial F}{\partial x} = 2x - \lambda y^2 & (1) \\ 0 = \frac{\partial F}{\partial y} = 2y - 2\lambda xy & (2) \\ 0 = \frac{\partial F}{\partial \lambda} = -xy^2+54 & (3) \end{cases}$$

(1): $2x=\lambda y^2 \Rightarrow x = \frac{\lambda y^2}{2}$.

Substitute into (2): $0 = 2y - 2\lambda \cdot \frac{\lambda y^2}{2} \cdot y = 2y - \lambda^2 y^3$

$$\Rightarrow y=0, \; y^2 = \frac{2}{\lambda^2}.$$

When $y=0$, $x=0$, but $g(0,0) \neq 54$.

Substitute $y^2 = \frac{2}{\lambda^2}$ into (1):

$$2x = \lambda y^2 = \lambda \cdot \frac{2}{\lambda^2} = \frac{2}{\lambda} \Rightarrow x = \frac{1}{\lambda}.$$

Substitute into (3):

$$\frac{1}{\lambda} \cdot \frac{2}{\lambda^2} = 54 \iff \lambda^3 = \frac{1}{27} \iff \lambda = \frac{1}{3}. \Rightarrow x = 3$$

$$y^2 = 18 \Rightarrow y = \pm 3\sqrt{2}.$$

$$f(3, 3\sqrt{2}) = 3^2 + (3\sqrt{2})^2 = 9 + 18 = 27$$

$$f(3, -3\sqrt{2}) = 3^2 + (-3\sqrt{2})^2 = 9 + 18 = 27.$$

So points on $xy^2 = 54$ nearest to origin are $(3, \pm 3\sqrt{2})$.

13.8 **11. Rectangle of greatest area in an ellipse** Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $x^2/16 + y^2/9 = 1$ with sides parallel to the coordinate axes.

Sol'n: $f(x,y) = 4xy$, $g(x,y) = \frac{x^2}{16} + \frac{y^2}{9}$.

Then consider $F(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - 1)$.

$$= xy - \frac{\lambda x^2}{16} - \frac{\lambda y^2}{9} + \lambda$$

$$\vec{\nabla} F(x,y,\lambda) = 0 \Leftrightarrow \begin{cases} 0 = \frac{\partial F}{\partial x} = 4y - \frac{\lambda x}{8} & (1) \\ 0 = \frac{\partial F}{\partial y} = 4x - \frac{2\lambda y}{9} & (2) \\ 0 = \frac{\partial F}{\partial \lambda} = -\frac{x^2}{16} - \frac{y^2}{9} + 1 & (3) \end{cases}$$

(1): $y = \frac{\lambda x}{32}$

Substitute into (2): $4\cancel{x} = \frac{2\lambda \cdot \frac{\lambda x}{32}}{9} \Rightarrow \lambda^2 = 4 \cdot 9 \cdot 16$ (since $g(x,y) = 1$, $x \neq 0$).

$\Rightarrow \lambda = \pm 24$

$\lambda = 24$: (1) becomes $y = \frac{24}{32}x = \frac{3}{4}x$

Substitute into (3):

$$1 = \frac{x^2}{16} + \frac{\left(\frac{3}{4}x\right)^2}{9} = \frac{x^2}{16} + \frac{\frac{9}{16}x^2}{9} = \frac{x^2}{8}. \Rightarrow x = \pm 2\sqrt{2},$$

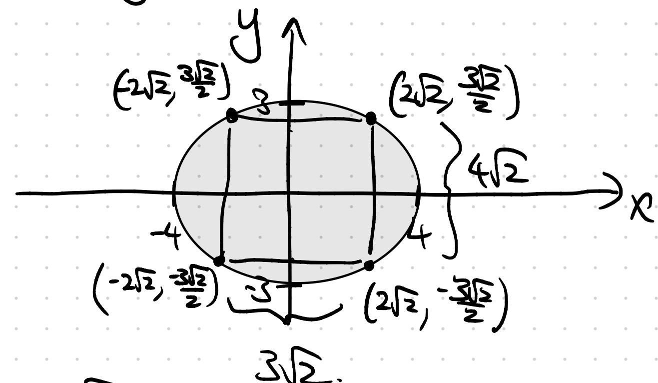
$$y = \pm \frac{3\sqrt{2}}{2}.$$

$\lambda = -24$: (1) becomes $y = -\frac{24}{32}x = \frac{-3}{4}x$

Substitute into (3):

$$1 = \frac{x^2}{16} + \frac{\left(-\frac{3}{4}x\right)^2}{9} = \frac{x^2}{8} \Rightarrow x = \pm 2\sqrt{2}$$

$$y = \mp \frac{3\sqrt{2}}{2}.$$



So area $= 2\sqrt{2} \cdot \frac{3\sqrt{2}}{2} = 12$ and length $= 4\sqrt{2}$

width $= 3\sqrt{2}$.

13.8

27. Rectangular box of largest volume in a sphere Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.

Sol'n: $f(x,y,z) = 8xyz \quad g(x,y,z) = x^2+y^2+z^2=1$

Consider $F(x,y,z,\lambda) = f(x,y,z) - \lambda(g(x,y,z)-1)$

$= 8xyz - \lambda x^2 - \lambda y^2 - \lambda z^2 + \lambda.$

$$\vec{\nabla}F(x,y,z,\lambda) = 0 \Leftrightarrow \begin{cases} 0 = \frac{\partial F}{\partial x} = 8yz - 2\lambda x & (1) \\ 0 = \frac{\partial F}{\partial y} = 8xz - 2\lambda y & (2) \\ 0 = \frac{\partial F}{\partial z} = 8xy - 2\lambda z & (3) \\ 0 = \frac{\partial F}{\partial \lambda} = -x^2 - y^2 - z^2 + 1 & (4) \end{cases}$$

(1)-(2): $0 = 8yz - 2\lambda x - 8xz + 2\lambda y = 8z(y-x) + 2\lambda(y-x) = (y-x)(8z+2\lambda)$

So either $y=x$ or $z=\frac{-\lambda}{4}$. Note can similarly obtain $y=x$ or $z=\frac{\lambda}{4}$.

Taking $z=\frac{-\lambda}{4}$ in (2) and (3) gives:

$0 = -2\lambda x - 2\lambda y = -2\lambda(x+y) \Rightarrow \lambda = 0$ or $x = -y$. If $\lambda = 0$, then $z = \frac{-\lambda}{4} = 0$, which we reject because of (4).

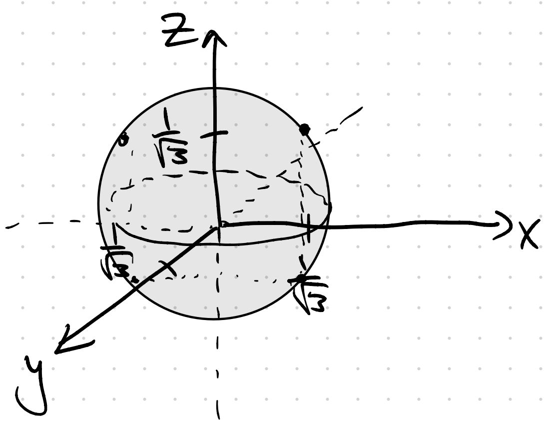
$0 = 8xy - 2\lambda \cdot \left(-\frac{\lambda}{4}\right) = 8xy + \frac{\lambda^2}{2}$.

$-x = y \curvearrowright = -8x^2 + \frac{\lambda^2}{2}$

$\Rightarrow 8x^2 = \frac{\lambda^2}{2} \Rightarrow x = \pm\frac{\lambda}{4}$. $\Rightarrow y = \pm\frac{\lambda}{4}$.

Then (4) becomes $0 = -\left(\frac{\pm\lambda}{4}\right)^2 - \left(\frac{\pm\lambda}{4}\right)^2 - \left(-\frac{\lambda}{4}\right)^2 + 1 = -\frac{3\lambda^2}{16} + 1 \Rightarrow \lambda^2 = \frac{16}{3} \Rightarrow \lambda = \pm\frac{4}{\sqrt{3}}$.

So $x = y = z = \pm\frac{1}{\sqrt{3}}$. So dimensions of box are $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$.



13.8 **47. a. Maximum on a sphere** Show that the maximum value of $a^2b^2c^2$ on a sphere of radius r centered at the origin of a Cartesian abc-coordinate system is $(r^2/3)^3$.

b. Geometric and arithmetic means Using part (a), show that for nonnegative numbers a, b, and c,

$$(abc)^{1/3} \leq \frac{a+b+c}{3};$$

that is, the *geometric mean* of three nonnegative numbers is less than or equal to their *arithmetic mean*.

Soln: a) $f(a,b,c) = a^2b^2c^2$. $g(a,b,c) = a^2+b^2+c^2 = r^2$.

$F(a,b,c) = f(a,b,c) - \lambda(g(a,b,c) - r^2)$.

$= a^2b^2c^2 - \lambda a^2 - \lambda b^2 - \lambda c^2 + \lambda r^2$

$$\vec{\nabla}F(a,b,c) = 0 \Leftrightarrow \begin{cases} 0 = \frac{\partial F}{\partial a} = 2ab^2c^2 - 2\lambda a & (1) \\ 0 = \frac{\partial f}{\partial b} = 2ba^2c^2 - 2\lambda b & (2) \\ 0 = \frac{\partial f}{\partial c} = 2ca^2b^2 - 2\lambda c & (3) \\ 0 = \frac{\partial F}{\partial \lambda} = -a^2 - b^2 - c^2 + r^2 & (4) \end{cases}$$

(1) − (2): $0 = 2ab^2c^2 - 2ba^2c^2 - 2\lambda a + 2\lambda b = 2abc^2(b-a) + 2\lambda(b-a)$

$= (b-a)(2abc^2 + 2\lambda)$

So either $a=b$, or $\lambda = -abc^2$.

When $a=b$, in (3): $0 = 2ca^4 - 2\lambda c = 2c(a^4 - \lambda)$

So either $c=0$ or $a^4 = \lambda$. $\Rightarrow$ $a = \lambda^{\frac{1}{4}}$

$b = \lambda^{\frac{1}{4}}$

but if $r \neq 0$, (4) means

$a, b, c \neq 0$.

(2) - (3): $0 = 2ba^2c^2 - 2ca^2b^2 - 2\lambda b + 2\lambda c = 2a^2bc(c-b) + 2\lambda(c-b)$

$= (c-b)(2a^2bc + 2\lambda)$.

So either $c=b$ or $\lambda = -a^2bc$.

So $c = \lambda^{\frac{1}{4}}$ as well.

Then in (4): $r^2 = 3(\lambda^{\frac{1}{4}})^2 = 3\lambda^{\frac{1}{2}}$, $\Leftrightarrow \lambda = \frac{r^4}{9}$, then $a = \frac{r}{\sqrt{3}}$, $b = \frac{r}{\sqrt{3}}$, $c = \frac{r}{\sqrt{3}}$.

and $f\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right) = \left(\frac{r^2}{3}\right)^3$ as required.

b) Part (a) gives

$$a^2 b^2 c^2 \leq \left(\frac{a^2+b^2+c^2}{3}\right)^3 \implies (a^2b^2c^2)^{1/3} \leq \frac{a^2+b^2+c^2}{3}$$

So when $x, y, z \geq 0$, we have that $x = a^2$ for some $a \in \mathbb{R}$
$y = b^2$ for some $b \in \mathbb{R}$
$z = c^2$ for some $c \in \mathbb{R}$

and we see from above that

$$(xyz)^{1/3} \leq \frac{x+y+z}{3}$$ as required. //

13.9

Finding Quadratic and Cubic Approximations

In Exercises 1–10, use Taylor's formula for $f(x, y)$ at the origin to find quadratic and cubic approximations of f near the origin.

9. $f(x, y) = \dfrac{1}{1 - x - y}$

From lecture notes: 2^{nd} order Taylor Polynomial of f at $\vec{a}$ in matrix form

$$P_2(\vec{x}) = f(\vec{a}) + \vec{\nabla} f(\vec{a})(\vec{x} - \vec{a}) + \tfrac{1}{2}(\vec{x} - \vec{a})^T Hf(\vec{a})(\vec{x} - \vec{a})$$

where $Hf(\vec{a}) = \begin{bmatrix} f_{x_1x_1}(\vec{a}) & \cdots & f_{x_1x_n}(\vec{a}) \\ \vdots & \ddots & \vdots \\ f_{x_nx_1}(\vec{a}) & \cdots & f_{x_nx_n}(\vec{a}) \end{bmatrix}$

$\vec{x} - \vec{a}$ are column vector

$\vec{\nabla} f$ row vector.

Soln: $f(0,0) = \dfrac{1}{1-0-0} = 1.$

$$\vec{\nabla} f(0,0) = \left[\frac{\partial f}{\partial x}\Big|_{(0,0)} \quad \frac{\partial f}{\partial y}\Big|_{(0,0)} \right]$$

$$\frac{\partial f}{\partial x}\Big|_{(0,0)} = \frac{(1-x-y)(0) - 1(-1)}{(1-x-y)^2}\Bigg|_{(0,0)} = \frac{1}{(1-x-y)^2}\Bigg|_{(0,0)} = 1.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \left.\frac{(1-x-y)(0)-1(-1)}{(1-x-y)^2}\right|_{(0,0)} = \left.\frac{1}{(1-x-y)^2}\right|_{(0,0)} = 1.$$

So $\vec{\nabla} f(0,0) = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$$(\vec{x}-\vec{0}) = \begin{bmatrix} x-0 \\ y-0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$f_{xx}(\vec{0}) = \left.\frac{(1-x-y)^2(0)-1(2(1-x-y)\cdot -1))}{(1-x-y)^4}\right|_{(0,0)} = \left.\frac{2(1-x-y)}{(1-x-y)^4}\right|_{(0,0)} = \left.\frac{2}{(1-x-y)^3}\right|_{(0,0)} = 2.$$

$$f_{xy}(\vec{0}) = \left.\frac{(1-x-y)^2(0)-1(2(1-x-y)-1))}{(1-x-y)^4}\right|_{(0,0)} = \left.\frac{2}{(1-x-y)^3}\right|_{(0,0)} = 2$$

$$f_{yx}(\vec{0}) = \left.\frac{2}{(1-x-y)^3}\right|_{(0,0)} = 2.$$

$$f_{yy}(\vec{0}) = \left.\frac{2}{(1-x-y)^3}\right|_{(0,0)} = 2.$$

$$\vec{\nabla} f(\vec{0}) \cdot (\vec{x}-\vec{0}) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x+y$$

$$Hf(\vec{0}) \cdot (\vec{x}-\vec{0}) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+2y \\ 2x+2y \end{bmatrix}$$

$$\frac{1}{2}(\vec{x}-\vec{0})^T Hf(\vec{0})(\vec{x}-\vec{0}) = \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2x+2y \\ 2x+2y \end{bmatrix} = \frac{1}{2}\left(x(2x+2y) + y(2x+2y)\right)$$

$$= \frac{1}{2}\left(2x^2+2xy+2xy+2y^2\right) = x^2+2xy+y^2.$$

So $P_2(\vec{x}) = 1 + x+y + x^2+2xy+y^2.$ //.