13.5

In Exercises 41–44, find a parametric equation for the line that is perpendicular to the graph of the given equation at the given point.

41. $x^2 + y^2 = 25, \quad (-3, 4)$

42. $x^2 + xy + y^2 = 3, \quad (2, -1)$

43. $x^2 + y^2 + z^2 = 14, \quad (3, -2, 1)$

Sol'n: Normal line to $f(x,y,z) = x^2+y^2+z^2 = 14$.

$r(t) = (x_0 + f_x(P_0)t,\ y_0 + f_y(P_0)t,\ z_0 + f_z(P_0)t)$.

$f_x(P_0) = 2x\big|_{(3,-2,1)} = 6, \quad f_y(P_0) = 2y\big|_{(3,-2,1)} = -4$

$f_z(P_0) = 2z\big|_{(3,-2,1)} = 2.$

So normal line is given by

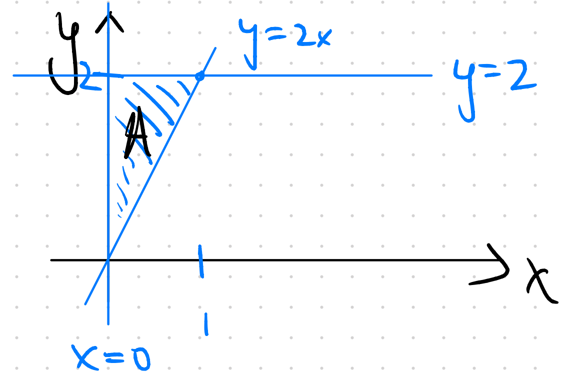
$r(t) = (3+6t,\ -2-4t,\ 1+2t),\quad t \in \mathbb{R}.$

13.7

Finding Absolute Extrema

In Exercises 31–38, find the absolute maxima and minima of the functions on the given domains.

31. $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant



Sol'n: Critical points:

f polynomial $\Rightarrow \vec{\nabla} f$ always exist.

$\vec{\nabla} f = (2x-4, 2y-4)$. So $\vec{0} = \vec{\nabla} f = (4x-4, 2y-4)$.

$\Rightarrow 4x-4 = 0 \Rightarrow x = 1$

$2y-4 = 0 \Rightarrow y = 2$.

and $f(1,2) = 2\cdot 1^2 - 4\cdot 1 + 2^2 - 4\cdot 2 + 1 = -5$.

Study f on bdry:

$f(0,y) = y^2 - 4y + 1$ and $\frac{df}{dy}\Big|_{x=0} = 2y-4$.

$\left.\frac{df}{dy}\right|_{x=0} = 0 \iff y = 2$ and $f(0,2) = 2^2 - 4\cdot 2 + 1 = 4 - 8 + 1 = -3.$

$f(x,2) = 2x^2 - 4x + 2^2 - 4\cdot 2 + 1 = 2x^2 - 4x - 3.$

$\left.\frac{df}{dx}\right|_{y=2} = 4x - 4$ and $\left.\frac{df}{dx}\right|_{y=2} = 0 \iff x = 1.$

$f(1,2) = -5$

$f(x,2x) = 2x^2 - 4x + (2x)^2 - 4(2x) + 1 = 6x^2 - 12x + 1.$

$\left.\frac{df}{dx}\right|_{y=2x} = 12x - 12.$ So $\left.\frac{df}{dx}\right|_{y=2x} = 0 \iff x = 1.$

$f(1,2) = -5$

$f(0,0) = 1.$ So absolute max at $(0,0)$ w/ $f(0,0) = 1$
absolute min at $(1,2)$ w/ $f(1,2) = -5.$

13.7 **39.** Find two numbers a and b with $a \leq b$ such that

$$\int_a^b (6 - x - x^2)\,dx$$

has its largest value.

Soln: $F(a,b) = \int_a^b (6-x-x^2)\,dx = \left(6x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_a^b$

$$= 6b - \frac{1}{2}b^2 - \frac{1}{3}b^3 - 6a + \frac{1}{2}a^2 + \frac{1}{3}a^3.$$

$\vec{\nabla} F = \left(-6 + a + a^2,\ 6 - b - b^2\right)$

and $\vec{\nabla} F = 0 \iff \begin{cases} -6 + a + a^2 = 0 \\ 6 - b - b^2 = 0 \end{cases} \iff \begin{cases} a = -3, 2 \\ b = -3, 2 \end{cases}$

$F(-3,-3) = 0 = F(2,2).$

Need $a \leq b$, so max at $(-3,2)$ w/ value $F(-3,2) = \boxed{\frac{125}{6}}$

13.7 In Exercises 63–66, find the absolute maximum and minimum values of the following functions on the given curves.

63. Functions:

a. $f(x, y) = x + y$ b. $g(x, y) = xy$

c. $h(x, y) = 2x^2 + y^2$

Curves:

i) The semicircle $x^2 + y^2 = 4$, $y \geq 0$

ii) The quarter circle $x^2 + y^2 = 4$, $x \geq 0$, $y \geq 0$

Use the parametric equations $x = 2\cos t$, $y = 2\sin t$.

Sol'n: a) i) On the bdry $f(2\cos t, 2\sin t) = 2\cos t + 2\sin t$. $0 \leq t \leq \pi$.

Min at $t = \pi$: $f = -2$.

$\frac{df}{dt} = -2\sin t + 2\cos t = 0 \iff \cos t = \sin t \iff t = \frac{\pi}{4}, \frac{3\pi}{4}$. $(y \geq 0)$.

$f(2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}) = 2\cos\frac{\pi}{4} + 2\sin\frac{\pi}{4} = \cancel{2}\frac{\sqrt{2}}{\cancel{2}} + \cancel{2}\frac{\sqrt{2}}{\cancel{2}} = 2\sqrt{2}$.

$f(2\cos\frac{3\pi}{4}, 2\sin\frac{3\pi}{4}) = 2\cos\frac{3\pi}{4} + 2\sin\frac{3\pi}{4} = -2\frac{\sqrt{2}}{2} + 2\frac{\sqrt{2}}{2} = 0$.

So Max at $t = \frac{\pi}{4}$, $f(2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}) = 2\sqrt{2}$.

ii) $0 \leq t \leq \frac{\pi}{2}$. Max at $t = \frac{\pi}{4}$, $f(2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}) = 2\sqrt{2}$.

Min at $t = 0, \frac{\pi}{2}$: $f(2\cos 0, 2\sin 0) = 2$

$f(2\cos\frac{\pi}{2}, 2\sin\frac{\pi}{2}) = 2$.

b) i) $g(2\cos t, 2\sin t) = 4\cos t\sin t$, $0 \leq t \leq \pi$.

$\frac{dg}{dt} = -4\sin^2 t + 4\cos^2 t = 0 \iff \sin^2 t = \cos^2 t. \iff t = \frac{\pi}{4}, \frac{3\pi}{4}$.

max $g(2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}) = 4\cos\frac{\pi}{4}\sin\frac{\pi}{4} = 4 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = 2$.

min $g(2\cos\frac{3\pi}{4}, 2\sin\frac{3\pi}{4}) = 4\cos\frac{3\pi}{4}\sin\frac{3\pi}{4} = -4\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = -2$.

ii) $0 \leq t \leq \frac{\pi}{2}$: Max still at $t = \frac{\pi}{4}$, $g = 2$.

Min at $t = 0, \frac{\pi}{2}$. $g(2\cos 0, 2\sin 0) = 0$

$g(2\cos\frac{\pi}{2}, 2\sin\frac{\pi}{2}) = 0$.

c) i) $h(2\cos t, 2\sin t) = 2\cdot 4\cos^2 t + 4\sin^2 t = 8\cos^2 t + 4\sin^2 t,$ $0 \leq t \leq \pi.$

$= 4\cos^2 t + 4.$

$\frac{dh}{dt} = -8\sin t\cos t = -4\sin(2t) = 0. \iff t = 0, \frac{\pi}{2}, \pi.$

$h(2\cos 0, 2\sin 0) = 4\cos^2 0 + 4 = 4+4 = 8.$

$h(2\cos\frac{\pi}{2}, 2\sin\frac{\pi}{2}) = 4\cos^2\frac{\pi}{2} + 4 = 4.$

$h(2\cos\pi, 2\sin\pi) = 4\cos^2\pi + 4 = 8.$

Max at $t = 0, \pi$, $h = 8.$

Min at $t = \frac{\pi}{2}$, $h = 4.$

ii) $0 \leq t \leq \frac{\pi}{2}.$ Max at $t = 0$, $h = 8$

Min at $t = \frac{\pi}{2}$, $h = 4.$

13.7

n=3 case

67. Least squares and regression lines When we try to fit a line $y = mx + b$ to a set of numerical data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, we usually choose the line that minimizes the sum of the squares of the vertical distances from the points to the line. In theory, this means finding the values of m and b that minimize the value of the function

$$w = (mx_1 + b - y_1)^2 + \cdots + (mx_n + b - y_n)^2. \qquad (1)$$

(See the accompanying figure.) Show that the values of m and b that do this are

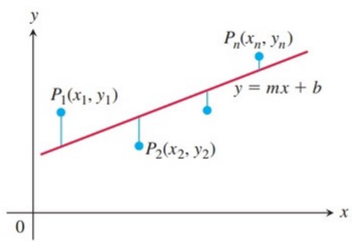
$$m = \frac{\left(\sum x_k\right)\left(\sum y_k\right) - n\sum x_k y_k}{\left(\sum x_k\right)^2 - n\sum x_k^2}, \qquad (2)$$

$$b = \frac{1}{n}\left(\sum y_k - m\sum x_k\right), \qquad (3)$$

with all sums running from $k = 1$ to $k = n$. Many scientific calculators have these formulas built in, enabling you to find m and b with only a few keystrokes after you have entered the data.

The line $y = mx + b$ determined by these values of m and b is called the **least squares line**, **regression line**, or **trend line** for the data under study. Finding a least squares line lets you

1. summarize data with a simple expression,
2. predict values of y for other, experimentally untried values of x,
3. handle data analytically.



n=3 case:

$$W(m,b) = \sum_{k=1}^{3} (mx_k + b - y_k)^2.$$

$$\frac{\partial w}{\partial m} = \sum_{k=1}^{3} 2(mx_k + b - y_k)x_k$$

$$\frac{\partial w}{\partial b} = \sum_{k=1}^{3} 2(mx_k + b - y_k)$$

So $\vec{\nabla} w = 0$

$$\Leftrightarrow \begin{cases} \sum_{k=1}^{3} 2(mx_k + b - y_k)x_k = 0 & (1) \\ \sum_{k=1}^{3} 2(mx_k + b - y_k) = 0 & (2). \end{cases}$$

So (2): $\cancel{2}mx_1 + \cancel{2}b - \cancel{2}y_1 + \cancel{2}mx_2 + \cancel{2}b - \cancel{2}y_2 + \cancel{2}mx_3 + \cancel{2}b - \cancel{2}y_3 = 0$

$$\Leftrightarrow \quad 3b = (y_1+y_2+y_3) - m(x_1+x_2+x_3)$$

$$\Rightarrow \quad b = \frac{1}{3}\left[(y_1+y_2+y_3) - m(x_1+x_2+x_3)\right]$$

(1): $\cancel{2}mx_1^2 + \cancel{2}bx_1 - \cancel{2}x_1y_1 + \cancel{2}mx_2^2 + \cancel{2}bx_2 - \cancel{2}x_2y_2 + \cancel{2}mx_3^2 + \cancel{2}bx_3 - \cancel{2}x_3y_3 = 0.$

$$m(x_1^2+x_2^2+x_3^2) + b(x_1+x_2+x_3) - x_1y_1 - x_2y_2 - x_3y_2 = 0.$$

$$\Rightarrow m(x_1^2+x_2^2+x_3^2) + \frac{1}{3}\left[(y_1+y_2+y_3) - m(x_1+x_2+x_3)\right](x_1+x_2+x_3) - x_1y_1 - x_2y_2 - x_3y_3 = 0$$

$$\Rightarrow m(x_1^2+x_2^2+x_3^2) - \frac{m}{3}(x_1+x_2+x_3)^2 = x_1y_1 + x_2y_2 + x_3y_3 - \frac{1}{3}(x_1+x_2+x_3)(y_1+y_2+y_3)$$

$$\Rightarrow m\left((x_1^2+x_2^2+x_3^2) - \frac{1}{3}(x_1+x_2+x_3)^2\right) = x_1y_1 + x_2y_2 + x_3 - \frac{1}{3}(x_1+x_2+x_3)(y_1+y_2+y_3)$$

$$m = \frac{\sum_{k=1}^{3} x_k y_k - \frac{1}{3}\left(\sum_{k=1}^{3} x_k\right)\left(\sum_{k=1}^{3} y_k\right)}{\left(\sum_{k=1}^{3} x_k^2\right) - \frac{1}{3}\left(\sum_{k=1}^{3} x_k\right)^2} \cdot \frac{-3}{-3}$$

$$= \frac{\left(\sum_{k=1}^{3} x_k\right)\left(\sum_{k=1}^{3} y_k\right) - 3\sum_{k=1}^{3} x_k y_k}{\left(\sum_{k=1}^{3} x_k\right)^2 - 3\left(\sum_{k=1}^{3} x_k^2\right)}$$