Second Fundmental Form

August 22, 2024

Clive Chan

Suppose $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$ is a regular surface on some open domain D.

Basic idea of open set: for every point inside, we can draw a small ball surrounding the point which lies entirely in the open set. This is useful, because when we do first principle for differentiation at a point p, we approach p from all possible directions; openness simply allows this to make sense. In particular, we just want to avoid this scenario: let f be a function defined on $[0,1]$, when we do first principle at f:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

at $x = 0$, $h \to 0^-$ is strange because f is not defined on the negative numbers.

Basic steps to find the second fundamental form, II:

1. Calculate Φ_u, Φ_v.
2. Calculate $\Phi_u \times \Phi_v$.
3. Calculate $||\Phi_u \times \Phi_v||$.
4. Calculate $\vec{n} = \frac{\Phi_u \times \Phi_v}{||\Phi_u \times \Phi_v||}$
5. Use your brain to judge whether $\Phi_{uu}, \Phi_{uv}, \Phi_{vv}$ or $\vec{n}_u, \vec{n}_v$ are easier to calculate.
6. The second fundamental form is

$$II = \begin{pmatrix} \Phi_{uu} \cdot \vec{n} & \Phi_{uv} \cdot \vec{n} \\ \Phi_{vu} \cdot \vec{n} & \Phi_{vv} \cdot \vec{n} \end{pmatrix} = -\begin{pmatrix} \Phi_u \cdot \vec{n}_u & \Phi_u \cdot \vec{n}_v \\ \Phi_v \cdot \vec{n}_u & \Phi_v \cdot \vec{n}_v \end{pmatrix}$$

Gauss curvature: $K = \frac{\det II}{\det I}$.

Mean curvature: $H = \frac{1}{2} tr II \cdot I^{-1}$.

Calculations for basic surfaces: watch my YouTube channel.

Question: why I^{-1}?

Answer: We want to express $d\vec{n}_p$ (opposite sign from II) in $\{\Phi_u, \Phi_v\}$. The matrix representation is $II \cdot I^{-1}$. (Proposition 3.4.9).