EPYMT TDG Group 2 Tutorial 3

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1 Space curve

Usually, we will try to parameterize the curve using one variable only.

Straight line: Just find the difference between two vectors:

Example: Parameterize the straight line joining $(1, 2, 3)$ and $(4, 5, 6)$, and with them as end-points:

Difference between two vectors: $(4, 5, 6) - (1, 2, 3) = (3, 3, 3)$.

Then take the point being subtracted as starting point (This time: $(1, 2, 3)$) and multiply the difference $(1, 2, 3)$ by a variable t such that $0 \leq t \leq 1$:

The line is $\{(1, 2, 3) + t(3, 3, 3) : 0 \leq t \leq 1\}$

Of course, you can choose $(4, 5, 6)$ as starting point, but this time the range of t would be $-1 \leq t \leq 0$ to ensure that $(1, 2, 3)$ will be generated by the non-zero end-point of the interval:

The line is $\{(4, 5, 6) - t(3, 3, 3) : -1 \leq t \leq 0\}$

Conic section is an important topic related to parametrization:

To begin with, let's recall some important trigometric identities and definition:

i $\cos^2 t + \sin^2 t = 1$.

ii $1 + \tan^2 t = \sec^2 t$.

iii $1 + \cot^2 t = \csc^2 t$.

iv $\sin 2t = 2 \sin t \cos t$.

v $\cos 2t = 1 - 2\sin^2 t = 2\cos^2 t - 1 = \cos^2 t - \sin^2 t$.

vi $\sinh t = \dfrac{e^t - e^{-t}}{2}$.

vii $\cosh t = \dfrac{e^t + e^{-t}}{2}$.

viii $\cosh^2 t - \sinh^2 t = 1$.

Example 1.1. *Circle: $x^2 + y^2 = r^2$ for some $r > 0$*

This is similar to (i) in the above identities.

Then let $x = r\cos t$ and $y = r\sin t$, we have

$$x^2 + y^2 = (r\cos t)^2 + (r\sin t)^2 = r^2(\cos^2 t + \sin^2 t) = r^2.$$

So we can parameterize circle by a variable t: $\{(r\cos t, r\sin t) \;\; 0 \le t < 2\pi\}$.

(As we want $(1,0)$ as the starting point of the path of a circle, we set $x = \cos t$ so that we have $\cos 0 = 1$.

Example 1.2. *Ellipse:* $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ *for some* $a, b > 0$

This is similar to (i) in the above identities, but we need some adjustment-to eliminate a and b in the L.H.S:

Let $x = a\cos t$ and $y = b\sin t$, we have

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{(a\cos t)}{a}\right)^2 + \left(\frac{(b\sin t)}{b}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

So we can parameterize circle by a variable t: $\{(a\cos t, b\sin t) \;\; 0 \le t < 2\pi\}$

Example 1.3. *Parabola:* $y = ax^2$

We can parameterize the curve by (t, at^2).

Actually, for a differentiable function $y = f(x)$, $\underline{\gamma(t) = (t, f(t))}$ is a natural parametrization of the curve.

Example 1.4. *Hyperbola:* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We can use either (ii) or (iv) to parameterize:

Let $(x, y) = (a\sec t, b\tan t)$ or $(a\cosh t, b\sinh t)$. Then

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{(a\sec t)^2}{a^2} - \frac{(b\tan t)^2}{b^2} = \sec^2 t - \tan^2 t = 1.$$

or

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{(a\cosh t)^2}{a^2} - \frac{(b\sinh t)^2}{b^2} = \cosh^2 t - \sinh^2 t = 1.$$

So we can parameterize hyperbola by the following two ways:

i *For region $x > 0$:* $\{(a\sec t, b\tan t) : -\frac{\pi}{2} < t < \frac{\pi}{2}\}$ *or* $\{(a\cosh t, b\sinh t) : \; t \in \mathbb{R}\}$

ii *For $x < 0$:* $\{(-a\sec t, b\tan t) : -\frac{\pi}{2} < t < \frac{\pi}{2}\}$, $\{(a\sec t, b\tan t) : \frac{\pi}{2} < t < \frac{3\pi}{2}\}$ *or* $\{(-a\cosh t, b\sinh t) : \; t \in \mathbb{R}\}$

Exercise 1.1. Parameterize the following curves:

i The line segment joining $(1, -2)$ and $(-3, 2)$.

ii The circle of radius 5 centered at $(3, -1)$.

iii The ellipse with equation $\frac{(x-1)^2}{4} + \frac{y^2}{9} = 1$.

iv The hyperbola $\frac{3x^2}{4} - \frac{12y^2}{13} = 2$ when $x > 0$

2 Computing Arc-length

Definition 2.1 (Regular parametrized curve)**.** *A regular parametrized curve is a differentiable function $\gamma : (a, b) \to \mathbb{R}^n$, $n = 2$ or 3, such that $\gamma'(t) \ne \mathbf{0}$. for any $t \in (a, b)$.*

Why do we need to care about regularity:

i Requiring the derivative to be non-zero enable us to find an inverse for the function.

What happen if the derivative equals to 0? Let's consider $f(x) = x^2$. Can we find an inverse in a neighborhood of $x = 0, y = 0$?

ii To make the curve to be "smooth" (No corners.) (e.g. $\gamma(t) = (t^2, t^3)$. Then $\gamma'(t) = (2t, 3t^2)$. When $t = 0$, we have $\gamma'(0) = (0, 0)$)

Definition 2.2 (Arc-length)**.** *Let $\gamma : (a, b) \to \mathbb{R}^n$ be a regular parametrized curve. Then the arc length of γ is defined by*

$$\ell(t) = \int_a^t \|\gamma'(u)\| \, du$$

In general, to ensure the integrability of $\|\gamma'(u)\|$, we will assume it to be continuous/piecewise continuous.

Remark (This remark will **definitely not appear** in tests/exams)*.* In general, regularity is not a must for the arc-length to be well-defined. More issues about points of continuity, like the "number" of discontinuity points/"size" of set of discontinuity points will be considered for integrability. Please refer to Mathematical Analysis textbook for more details.

Note that this definition is independent of parametrization:
E.g. If $\alpha : (c, d) \to \mathbb{R}^n$ represent the same curve as γ: i.e. $\alpha(c) = \gamma(a)$, $\alpha(d) = \gamma(b)$, and there exists a differentiable bijective map $\phi : (a, b) \to (c, d)$ such that $\gamma(t) = \alpha(\phi(t))$ and $\phi'(t) > 0$ on $a \le t \le b$. Then we have

$$\begin{aligned}
\text{Arc-length computed by } \gamma &= \int_a^b \|\gamma'(t)\| \, dt \\
&= \int_a^b \|[\alpha(\phi(t))]'\| \, dt \\
&= \int_a^b \|[\alpha'(\phi(t))]\phi'(t)\| \, dt \\
&= \int_a^b \|[\alpha'(\phi(t))]\|\phi'(t) \, dt \text{ (As we assume } \phi'(t) > 0 \text{ on } a \le t \le b) \\
&= \int_c^d \|\alpha'(u)\| \, du \text{ (By letting } u = \phi(t), \text{ then } du = \phi'(t),\ \phi(a) = c, \phi(b) = d) \\
&= \text{Arc-length computed by } \alpha
\end{aligned}$$

Examples and Exercises:

Example 2.1. *Part of Cycloid: $\gamma(\theta) = (\theta - \sin\theta, 1 - \cos\theta)$, $0 < \theta < 2\pi$.*
Note that it is regular in the given range:

$$\gamma'(\theta) = (1 - \cos\theta, \sin\theta)$$
$$\textit{If } \gamma'(\theta) \neq 0, \textit{ we have} \cos\theta \neq 1 \textit{ and } \sin\theta \neq 0$$
$$\theta \neq 0, 2\pi$$

(So we remove the end-point to make it to be regular. Also note that the removal of finite number of points will not affect the integral.) Then the required arclength will be :

$$
\begin{aligned}
\ell &= \int_0^{2\pi} \|\gamma'(\theta)\| \; d\theta \\
&= \int_0^{2\pi} \|(1-\cos\theta, \sin\theta)\| \; d\theta \\
&= \int_0^{2\pi} \sqrt{[(1-\cos\theta)^2 + \sin^2\theta]} \; d\theta \\
&= \int_0^{2\pi} \sqrt{[2-2\cos\theta]} \; d\theta \\
&= \int_0^{2\pi} \sqrt{[2-2(1-2\sin^2\frac{\theta}{2})]} \; d\theta \\
&= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}} \; d\theta \\
&= 2\int_0^{2\pi} \sin\frac{\theta}{2} \; d\theta \; \textit{(As } 0 < \theta < 2\pi, \textit{ we have } 0 < \frac{\theta}{2} < \pi, \textit{ which means } \sin\frac{\theta}{2} \geq 0\textit{)} \\
&= 4[-\cos\frac{\theta}{2}]_0^{2\pi} \\
&= -4[-1-1] \\
&= 8
\end{aligned}
$$

Remark. In the integration process, we try to eliminate "1" from $1-\cos\theta$ bye double angle formula:

$$
\begin{aligned}
1+\cos\theta &= 1 + 2\cos^2\frac{\theta}{2} - 1 \\
&= 2\cos^2\frac{\theta}{2} \\
1-\cos\theta &= 1 - \left(1 - 2\sin^2\frac{\theta}{2}\right) \\
&= 2\sin^2\frac{\theta}{2}
\end{aligned}
$$

Besides, Note that we can use similar method to compute the arclength of cycloid when $2\pi < \theta < 4\pi$. However, note that

$$
\int_{2\pi}^{4\pi} \sqrt{4\sin^2\frac{\theta}{2}} = -2\int_0^{2\pi} \sin\frac{\theta}{2} \; d\theta
$$

as $\pi < \frac{\theta}{2} < 2\pi$ and $\sin\frac{\theta}{2} \leq 0$ here.
You need to be very aware of whether $\sqrt{f^2(x)} = f(x)$ in given range of x. As I mention in the tutorial, $\sqrt{x} \geq 0$ for all $x \in \mathbb{R}$.

$$
\begin{aligned}
&\text{If } f(x) \geq 0, \text{ we have } \sqrt{f^2(x)} = f(x) \\
&\text{If } f(x) \geq 0, \text{ we have } \sqrt{f^2(x)} = -f(x)
\end{aligned}
$$

Exercise 2.1. Compute the arclength of the following curves:

i $\quad y = \dfrac{x^4+3}{6x}$ from $x = 1$ to $x = 2$.

ii The deltoid parametrized by $\mathbf{r}(\theta) = (2\cos\theta + \cos 2\theta, 2\sin\theta - \sin 2\theta)$, $0 \le \theta \le 2\pi$.

iii The astroid defined by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

3 Arc Length parametrization and computation

Key: Parametrization is not unique:

Recall:

Parametrization of unit circle ($x^2 + y^2 = 1$): $\{(\cos t, \sin t)\ \ 0 \le t < 2\pi\}$

How about $\{(\cos 2t, \sin 2t)\ \ 0 \le t < 2\pi\}$? Is this still a circle? What is the difference between these two parametrization?

i (1) $\{(\cos t, \sin t)\ \ 0 \le t < 2\pi\}$: Passes through each point on the circle **once**.

(2) $\{(\cos 2t, \sin 2t)\ \ 0 \le t < 2\pi\}$: Passes through each point on the circle **twice**.

ii The arc-length computed on the whole domain is different:

(1) $\{(\cos t, \sin t)\ \ 0 \le t < 2\pi\}$: $\displaystyle\int_0^{2\pi} \|(-\sin t, \cos t)\| = 2\pi$.

(2) $\{(\cos 2t, \sin 2t)\ \ 0 \le t < 2\pi\}$: $\displaystyle\int_0^{2\pi} \|(-2\sin t, 2\cos t)\| = 4\pi$.

Also, note that for the first parametrization, the arc-length computed is just the difference of two endpoints. That's where arc-length parametrization comes from: To make $\|\gamma'(t)\| = 1$ such that the **arc-length is just the difference of two end-points**.

Steps to parameterize $\gamma(t) : [a, b] \to \mathbb{R}^n$:

Step 1: Let $s(t) = \displaystyle\int_a^t \|\gamma'(u)\|\, du$. Note that after integration, R.H.S is a function in terms of t only.

Step 2: Make t as the subject of the formula of $s(t) = ...$

Step 3: Then we take $\gamma(s) = \gamma(t(s))$ as a arc-length parameterized curve in terms of s(here s can be regarded as a variable instead of a function.)

Idea: Note that we set up one more function $t(\cdot)$ as we want to use chain rule to have

$$\left\|\frac{d\gamma(t)}{d\cdot}\right\| = \left\|\frac{d\gamma(t)}{dt}\right\| \left\|\frac{dt}{d\cdot}\right\| = 1$$

Hence we want to find $t(\cdot)$ such that

$$\left\|\frac{dt}{d\cdot}\right\| = \frac{1}{\|\gamma(t)\|}$$

, which only makes sense when $\gamma(t)$ is a regular path (i.e. $\|\gamma(t)\| \neq 0$ for all $t \in [a, b]$)

We then have

$$\left\|\frac{d\cdot}{dt}\right\| = \|\gamma(t)\|.$$

And note that differentiation and integration cancel each other, so $s'(t) = \|\gamma'(t)\|$, then we can take $\cdot = s$, and we can use inverse function theorem to express t in terms of s.

Example 3.1. *Find the arclength parametrization of the curve $\gamma : [a, \infty) \to \mathbb{R}^2$, denoted by $\gamma(t) = (3\cos 2t, 3\sin 2t)$.*
Then

$$\begin{aligned} s(t) &= \int_a^t \|(-6\sin 2u, 6\cos 2u)\| \ du \\ &= 6\int_a^t \sqrt{\sin^2 2u + \cos^2 2u} \ du \\ &= 6(t-a) \end{aligned}$$

Then we make t to be subject of the formula:

$$t = \frac{s}{6} + a$$

Then we take $\gamma(s) = \left(3\cos\left[2\cdot\left(\frac{s}{6}+a\right)\right], 3\sin\left[2\cdot\left(\frac{s}{6}+a\right)\right]\right) = \left(3\cos\left[\frac{s}{3}+2a\right], 3\sin\left[\frac{s}{3}+2a\right]\right)$.
Then note that

$$\begin{aligned} \|\gamma'(s)\| &= \| \left(-\sin\left[\frac{s}{3}+2a\right], \cos\left[\frac{s}{3}+2a\right]\right) \| \\ &= 1 \end{aligned}$$

Example 3.2 (Logarithmic spiral)**.** *Find the arc length parametrization of the curve $\gamma : [0, \infty) \to \mathbb{R}^2$, denoted by $\gamma(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$, where $a > 0, b < 0$.*
Then

$$\begin{aligned} s(t) &= \int_0^t \|\gamma'(u)\| \ du \\ &= \int_0^t \|(abe^{bu}\cos u - ae^{bu}\sin u, abe^{bu}\sin u + ae^{bu}\cos u)\| \ du \\ &= a\int_0^t \sqrt{b^2e^{2bu}\cos^2 u - 2be^{2bu}\sin u\cos u + e^{2bu}\sin^2 u + b^2e^{2bu}\sin^2 u + 2be^{2bu}\sin u\cos u + e^{2bu}\cos^2 u} \ du \\ &= a\int_0^t \sqrt{b^2e^{2bu}[\cos^2 u + \sin^2 u] + e^{2bu}[\sin^2 u + \cos^2 u]} \ du \\ &= a\int_0^t \sqrt{b^2e^{2bu} + e^{2bu}} \ du \\ &= a\sqrt{b^2+1}\int_0^t e^{bu} \ du \\ &= \frac{a\sqrt{b^2+1}}{b}[e^{bu}]_0^t \\ &= \frac{a\sqrt{b^2+1}}{b}(e^{bt}-1) \end{aligned}$$

Then by changing t to be the subject of the formula, we have

$$t = \frac{\ln\left[\dfrac{bs}{a\sqrt{b^2+1}} + 1\right]}{b}$$

Then we have

$$\begin{aligned}\gamma(s) &= (ae^{bt(s)}\cos t(s), ae^{bt(s)}\sin t(s)) \\ &= \left(a\left[\frac{bs}{a\sqrt{b^2+1}}+1\right]\cos\left(\frac{\ln\left[\frac{bs}{a\sqrt{b^2+1}}+1\right]}{b}\right), a\left[\frac{bs}{a\sqrt{b^2+1}}+1\right]\sin\left(\frac{\ln\left[\frac{bs}{a\sqrt{b^2+1}}+1\right]}{b}\right)\right)\end{aligned}$$

to be the curve parameterized by arclength.

Exercise 3.1 (Related question to logarithmic spiral, adapted from 2021 TDG Quiz 1 Q5 with slight modification)**.** Let $\mathbf{x}(\theta) : \theta \in [0, \infty) \to \mathbb{R}^3$ be a regular curve. Denote the partial arc-length of $\mathbf{x}(\theta)$ by

$$\ell(\theta) = \int_0^\theta \|\mathbf{x}'(u)\|\, du.$$

i Show that $\ell(\theta)$ is an **injective** function.

ii The **generalized** logarithmic spiral $\mathbf{y}(\theta) : \theta \in [0, \infty) \to \mathbb{R}^3$ is defined by

$$\mathbf{y}(\theta) = (e^{-\theta}\cos\theta, e^{-\theta}\sin\theta, \lambda\theta),$$

where $\lambda \neq 0$.

(1) Show that if $b < 0$, then $\mathbf{x}(\theta) = (ae^{b\theta}\cos\theta, ae^{b\theta}\sin\theta)$ has **finite** arc-length, that is, $\ell(\infty) = \lim_{\theta\to\infty}\int_0^\theta \|\mathbf{x}'(u)\|\, du$ exists and is finite.

(2) Does $\mathbf{y}(\theta)$ have finite arc-length for $\lambda \neq 0$? Explain your answer.

Exercise 3.2.

i It is given that the following curves are parametrized by arc-length. Find the value of p where $p > 0$.

(1) $\mathbf{r}(\theta) = (4\sin p\theta, -4\cos p\theta, 3p\theta)$

(2) $\mathbf{r}(\theta) = (p\cos\theta, 2+\sin\theta, 1-\frac{\sqrt{3}}{2}\cos\theta)$, for $0 < \theta < 2\pi$.

(3) $\mathbf{r}(t) = (\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1-t)^{\frac{3}{2}}, pt)$ for $0 < t < 1$.

ii Express the following curve in the form that is parameterized by arc-length: Line joining $(2, 3, 4)$ and $(4, 2, 5)$.

iii (2012 TDG Quiz 1 Q3)
Let $f : [-4, -1] \to \mathbb{R}$ be defined by $f(t) = \dfrac{1}{2}(t\sqrt{t^2-1} - \ln(\sqrt{t^2-1}+t))$.

(1) Show that $f'(t) = \sqrt{t^2-1}$.

(2) Find a regular parametrization of the graph of f.

(3) Find an arc-length parametrization for this curve.

iv Prove that the arc length of the curve given by the graph of function $r = r(\theta)$, $\alpha < \theta < \beta$, in polar coordinates is

$$\ell = \int_\alpha^\beta \sqrt{r'^2 + r^2}\, d\theta.$$

4 Curve curvature (Will not appear in Test 1)

Intuitive idea of curvature: The rate of the "bending" of the curve away from its normal.

Intuitive idea of torsion: The rate of the "bending" of the curve away from its "tangent plane".

Recall that

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}.$$

Example on computation: (Just computational exhaustive):

Find the curvature of(Logarithmic spiral: $\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$. $a > 0, b > 0$:

From the above, we have computed

$$\alpha'(t) = (abe^{bt}\cos t - ae^{bt}\sin t, abe^{bt}\sin t + ae^{bt}\cos t).$$

and

$$\|\alpha'(t)\| = ae^{bt}\sqrt{1+b^2}.$$

Hence

$$\begin{aligned}\alpha''(t) &= (ab^2e^{bt}\cos t - abe^{bt}\sin t - abe^{bt}\sin t - ae^{bt}\cos t, ab^2e^{bt}\sin t + abe^{bt}\cos t + abe^{bt}\cos t - ae^{bt}\sin t)\\ &= (ab^2e^{bt}\cos t - 2abe^{bt}\sin t - ae^{bt}\cos t, ab^2e^{bt}\sin t + 2abe^{bt}\cos t - ae^{bt}\sin t)\end{aligned}$$

Therefore, we have

$$\begin{aligned}\alpha'(t) \times \alpha''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ abe^{bt}\cos t - ae^{bt}\sin t & abe^{bt}\sin t + ae^{bt}\cos t & 0 \\ ab^2e^{bt}\cos t - 2abe^{bt}\sin t - ae^{bt}\cos t & ab^2e^{bt}\sin t + 2abe^{bt}\cos t - ae^{bt}\sin t & 0 \end{vmatrix}\\ &= [(abe^{bt}\cos t - ae^{bt}\sin t)(ab^2e^{bt}\sin t + 2abe^{bt}\cos t - ae^{bt}\sin t)\\ &- (abe^{bt}\sin t + ae^{bt}\cos t)(ab^2e^{bt}\cos t - 2abe^{bt}\sin t - ae^{bt}\cos t)]\mathbf{k}\\ &= a^2e^{2bt}[b^3\sin t\cos t + 2b^2\cos^2 t - b\sin t\cos t - b^2\sin^2 t - 2b\sin t\cos t + \sin^2 t\\ &- b^3\sin t\cos t + 2b^2\sin^2 t - b\sin t\cos t + b^2\cos^2 t + 2b\sin t\cos t + \cos^2 t]\mathbf{k}\\ &= a^2e^{2bt}[b^2\cos^2 t + b^2a^2\sin^2 t + + \sin^2 t + \cos^2 t]\mathbf{k}\\ &= a^2(1+b^2)e^{2bt}\mathbf{k}\end{aligned}$$

Therefore we have

$$\begin{aligned}\kappa(t) &= \frac{\|\alpha'(t)\times\alpha''(t)\|}{\|\alpha'(t)\|^3}\\ &= \frac{a^2(1+b^2)e^{2bt}}{(ae^{bt}\sqrt{1+b^2})^3}\\ &= \frac{a^2(1+b^2)e^{2bt}}{a^3e^{3bt}(1+b^2)^{\frac{3}{2}}}\\ &= \frac{1}{ae^{bt}\sqrt{1+b^2}}\end{aligned}$$

Exercise: Prove that the curvature of the curve defined by $r=r(\theta)$ in polar coordinates is given by

$$\kappa(\theta) = \frac{|2r'^2 - rr'' + r^2|}{(r^2+r'^2)^{\frac{3}{2}}}$$

Key:

i Recall the formula $\kappa(\theta) = \dfrac{|x'y''-x''y'|}{(x'^2+y'^2)^{\frac{3}{2}}}$ for 2-D curve. What we need to do is to express the curve in terms of Catesterian coordinate first.

ii Note that for $r=r(\theta)$, the radial component r is a function of θ, the angular component (This is just the same as using f to express $f(x)$. Don't regard r in our question as a constant!).
Hence we can express the curve as $(r(\theta)\cos\theta, r(\theta)\sin\theta)$.

iii Then we have

$$\begin{aligned}x'(\theta) &= r'(\theta)\cos\theta - r(\theta)\sin\theta\\ y'(\theta) &= r'(\theta)\sin\theta + r(\theta)\cos\theta\\ x''(\theta) &= r''(\theta)\cos\theta - r'(\theta)\sin\theta - r'(\theta)\sin\theta - r(\theta)\cos\theta = r''(\theta)\cos\theta - 2r'(\theta)\sin\theta - r(\theta)\cos\theta\\ y''(\theta) &= r''(\theta)\sin\theta + r'(\theta)\cos\theta + r'(\theta)\cos\theta - r(\theta)\sin\theta = r''(\theta)\sin\theta + +2r'(\theta)\cos\theta - r(\theta)\sin\theta\end{aligned}$$

Then we have

$$\begin{aligned}[x'^2+y'^2]^{\frac{3}{2}} &= [(r'(\theta)\cos\theta - r(\theta)\sin\theta)^2 + (r'(\theta)\sin\theta + r(\theta)\cos\theta)^2]^{\frac{3}{2}}\\ &= [r'(\theta)^2\cos^2\theta - 2r(\theta)r'(\theta)\sin\theta\cos\theta + r(\theta)^2\sin^2\theta + r'(\theta)^2\sin^2\theta + 2r(\theta)r'(\theta)\sin\theta\cos\theta + r(\theta)^2\cos^2\theta]^{\frac{3}{2}}\\ &= [r'(\theta)^2(\cos^2\theta+\sin^2\theta) + r(\theta)^2(\cos^2\theta+\sin^2\theta)]^{\frac{3}{2}}\\ &= (r(\theta)^2 + r'(\theta)^2)^{\frac{3}{2}}\end{aligned}$$

$$\begin{aligned}x'y'' &= [r'(\theta)\cos\theta - r(\theta)\sin\theta][r''(\theta)\sin\theta + 2r'(\theta)\cos\theta - r(\theta)\sin\theta]\\ &= r'(\theta)r''(\theta)\sin\theta\cos\theta + 2r'(\theta)^2\cos^2\theta - r(\theta)r'(\theta)\sin\theta\cos\theta - r(\theta)r''(\theta)\sin^2\theta - 2r(\theta)r'(\theta)\sin\theta\cos\theta + r(\theta)^2\sin^2\theta\end{aligned}$$

$$
\begin{aligned}
x''y' &= [r''(\theta)\cos\theta - 2r'(\theta)\sin\theta - r(\theta)\cos\theta][r'(\theta)\sin\theta + r(\theta)\cos\theta] \\
&= r'(\theta)r''(\theta)\sin\theta\cos\theta + r(\theta)r''(\theta)\cos^2\theta - 2r'^2(\theta)\sin^2\theta - 2r(\theta)r'(\theta)\sin\theta\cos\theta - r(\theta)r'(\theta)\sin\theta\cos\theta - r(\theta)^2\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
x'y'' - x''y' &= 2r'(\theta)^2[\cos^2\theta + \sin^2\theta] - r(\theta)r''(\theta)[\sin^2\theta + \cos^2\theta] + r(\theta)^2[\cos^2\theta + \sin^2\theta] \\
&= 2r'(\theta)^2 - r(\theta)r''(\theta) + r(\theta)^2
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\kappa(\theta) &= \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{\frac{3}{2}}} \\
&= \frac{|2r'(\theta)^2 - r(\theta)r''(\theta) + r(\theta)^2|}{(r(\theta)^2 + r'(\theta)^2)^{\frac{3}{2}}}
\end{aligned}
$$

Exercise:

i Consider the curve C given by the graph of the function $y = \ln\csc x$, $0 < x < \pi$, in rectangular coordinates.

(1) Show that $\mathbf{r}(s) = (2\tan^{-1} e^s, \ln\cosh s)$, $s \in \mathbb{R}$ is an arc length parametrization of C.

(2) Show that the curvature of the curve is

$$
\kappa(s) = \frac{1}{\cosh s}
$$

5 Shortest Distance between a point and a plane

Steps of finding the shortest distance between a point A with position vector $\mathbf{a}$ and a plane spanned by $\{\mathbf{b}, \mathbf{c}\}$, where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^{\mathbf{3}}$:

Step 1: Compute $\mathbf{n} = \mathbf{b} \times \mathbf{c}$

Step 2: Find the projection of $\mathbf{a}$ onto $\mathbf{n}$ by the formula $\text{Proj}_{\mathbf{n}}\mathbf{a} = \dfrac{\langle \mathbf{a}, \mathbf{n} \rangle}{\|\mathbf{n}\|^{\mathbf{2}}}\mathbf{n}$

Step 3: Compute $\|\text{Proj}_{\mathbf{n}}\mathbf{a}\|$. This is the required shortest distance.

Example 5.1. *Compute the shortest distance between a point* $(1, 0, 0)$ *with position vector* $\mathbf{a}$ *and a plane spanned by* $\{(0, 1, 1), (1, 1, 0)\}$

Solution.

Step 1: *Compute* $\mathbf{n} = \mathbf{b} \times \mathbf{c}$*:*

$$\begin{aligned}
\mathbf{n} &= (0,1,1) \times (1,1,0) \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} \\
&= (-1,1,-1)
\end{aligned}$$

Step 2: *Find the projection of* $\mathbf{a}$ *onto* $\mathbf{n}$ *by the formula* $Proj_{\mathbf{n}}\mathbf{a} = \dfrac{\langle \mathbf{a}, \mathbf{n} \rangle}{\|\mathbf{n}\|^{2}}\mathbf{n}$*:*

$$\begin{aligned}
Proj_{\mathbf{n}}\mathbf{a} &= \frac{\langle (1,0,0), (-1,1,-1) \rangle}{1^2 + (-1)^2 + 1^2}(-1,1,-1) \\
&= \frac{-1}{3}(-1,1,-1) \\
&= \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)
\end{aligned}$$

Step 3: *Compute* $\|Proj_{\mathbf{n}}\mathbf{a}\|$*:*

Required shortest distance$= \sqrt{\left(\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \dfrac{\sqrt{3}}{3}.$

Example 5.2. *Compute the shortest distance between a point* $(1,2,3)$ *with position vector* $\mathbf{a}$ *and a plane spanned by* $\{(1,2,4),(2,3,1)\}$

Solution.

Step 1: *Compute* $\mathbf{n} = \mathbf{b} \times \mathbf{c}$*:*

$$\begin{aligned}
\mathbf{n} &= (1,2,4) \times (2,3,1) \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 4 \\ 2 & 3 & 1 \end{vmatrix} \\
&= (-10,7,-1)
\end{aligned}$$

Step 2: *Find the projection of* $\mathbf{a}$ *onto* $\mathbf{n}$ *by the formula* $Proj_{\mathbf{n}}\mathbf{a} = \dfrac{\langle \mathbf{a}, \mathbf{n} \rangle}{\|\mathbf{n}\|^{2}}\mathbf{n}$*:*

$$\begin{aligned}
Proj_{\mathbf{n}}\mathbf{a} &= \frac{\langle (1,2,3), (-10,7,-1) \rangle}{(-10)^2 + 7^2 + (-1)^2}(-10,7,-1) \\
&= \frac{1}{150}(-10,7,-1)
\end{aligned}$$

Step 3: *Compute* $\|Proj_{\mathbf{n}}\mathbf{a}\|$*:*

Required shortest distance$= \dfrac{1}{150}\sqrt{10^2 + 7^2 + 1^2} = \dfrac{1}{\sqrt{150}}.$

6 Revision

i (1) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Prove the polarization identity

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}\left(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2\right)$$

(2) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Prove that if $\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ for any $\mathbf{w} \in \mathbb{R}^3$, then $\mathbf{u} = \mathbf{v}$.

(3) Prove that for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, we have $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

(4) Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be any properly defined differentiable functions. Prove that

$$\frac{d}{dt}\langle \mathbf{x}(t), \mathbf{y}(t) \rangle \equiv \langle \frac{d}{dt}\mathbf{x}(t), \mathbf{y}(t) \rangle + \langle \mathbf{x}(t), \frac{d}{dt}\mathbf{y}(t) \rangle.$$

by first principle.

ii

iii 2021 TDG Quiz 1 Q4

Let $(V, <\cdot, \cdot>)$ be an n-dimensional inner product space over the real numbers with $n < \infty$.

(1) Show that for any **orthonormal basis** $\{\mathbf{e_i}\}_{\mathbf{i=1}}^{\mathbf{n}}$, it holds

$$\mathbf{x} = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{e_i} \rangle \mathbf{e_i} \text{ and } \sum_{i=1}^{m} |\langle \mathbf{x}, \mathbf{e_i} \rangle|^2 \leq \|x\|^2 \text{ for all integers } m \text{ such that } 1 \leq m \leq n.$$

(2) If $\{\mathbf{v_i}\}_{\mathbf{i=1}}^{\mathbf{n}}$ is an arbitrary basis for V , will the relation

$$\mathbf{x} = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v_i} \rangle \mathbf{v_i}$$

hold? If yes, give a proof; otherwise, raise a counterexample.

iv (Problem set 2 challenging question) Let $\gamma(s)$ be a differentiable vector-valued function on $\mathbb{R}^2$ with $\|\gamma'(s)\| = 1$. Denote $\mathbf{T}(s) = \gamma'(s)$

(a). Show that $\langle \mathbf{T}'(s), \mathbf{T}(s) \rangle = 0$

(b). Denote $\kappa(s) = \|\mathbf{T}'(s)\|$, define $\mathbf{N}(s)$ by the relation: $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$

Compute $\|\mathbf{N}(s)\|$ and $\langle \mathbf{T}(s), \mathbf{N}(s) \rangle$, deduce that $\langle \mathbf{T}(s), \mathbf{N}'(s) \rangle = -\kappa(s)$

(c). Does $\{\mathbf{T}(s), \mathbf{N}(s)\}$ constitute a orthonormal basis for $\mathbb{R}^2$? Hence prove or disprove:

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s)$$

Solution. *i (1)* *Suppose* $\mathbf{u} \neq \mathbf{v}$, *then we have* $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$.

Take $\mathbf{w} = \mathbf{u} - \mathbf{v}$. *Then by assumptions in the question, we have*

$$\begin{aligned} 0 &=< \mathbf{u}, \mathbf{w} > - < \mathbf{v}, \mathbf{w} > \\ &=< \mathbf{u} - \mathbf{v}, \mathbf{w} > \\ &=< \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} > \\ &= \|\mathbf{u} - \mathbf{v}\|^2 \\ &> 0 \text{ (As we assume } \mathbf{u} - \mathbf{v} \neq \mathbf{0}.) \end{aligned}$$

We have $0 > 0$. *Contradiction arises.*

Then we have $\mathbf{u} = \mathbf{v}$

(2) *Let* $\mathbf{u} = (u_1, u_2, u_3)$ *and* $\mathbf{v} = (v_1, v_2, v_3)$. *Then we have*

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

Then we have

$$\begin{aligned} < \mathbf{u}, \mathbf{u} \times \mathbf{v} > &=< (u_1, u_2, u_3), (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) > \\ &= u_1u_2v_3 - u_1u_3v_2 + u_3u_2v_1 - u_1u_2v_3 + u_1v_2u_3 - u_2v_1u_3 \\ &= 0 \end{aligned}$$

So we have $\mathbf{u}$ *orthogonal to* $\mathbf{u} \times \mathbf{v}$.

Similarly, we have $\mathbf{v}$ *orthogonal to* $\mathbf{u} \times \mathbf{v}$.

(3) *Simple proof:*

Note that $\mathbf{x}(\mathbf{t}) = (\mathbf{x_1}(\mathbf{t}), \mathbf{x_2}(\mathbf{t}), \mathbf{x_3}(\mathbf{t}))$ *and* $\mathbf{y}(\mathbf{t}) = (\mathbf{y_1}(\mathbf{t}), \mathbf{y_2}(\mathbf{t}), \mathbf{y_3}(\mathbf{t}))$ *for some differentiable function* $x_1(t), x_2(t), x_3(t), y_1(t), y_2(t), y_3(t)$.

Then note that

$$\begin{aligned} \frac{d}{dt}\langle \mathbf{x}(t), \mathbf{y}(t) \rangle &= \frac{d}{dt}[x_1(t)y_1(t) + x_2(t)y_2(t) + x_3(t)y_3(t)] \\ &= x_1'(t)y_1(t) + x_1(t)y_1'(t) + x_2'(t)y_2(t) + x_2(t)y_2'(t) + x_3'(t)y_3(t) + x_3(t)y_3'(t) \\ &= x_1'(t)y_1(t) + x_2'(t)y_2(t) + x_3'(t)y_3(t) + x_1(t)y_1'(t) + x_2(t)y_2'(t) + x_3(t)y_3'(t) \\ &= \langle \mathbf{x}'(t), \mathbf{y}(t) \rangle + \langle \mathbf{x}(t), \mathbf{y}'(t) \rangle \end{aligned}$$

Proof by first principle:

$$
\begin{aligned}
\frac{d}{dt}\langle \mathbf{x}(t), \mathbf{y}(t)\rangle &= \lim_{h\to 0} \frac{\langle \mathbf{x}(t+h), \mathbf{y}(t+h)\rangle - \langle \mathbf{x}(t), \mathbf{y}(t)\rangle}{h} \\
&= \lim_{h\to 0} \frac{\langle \mathbf{x}(t+h), \mathbf{y}(t+h)\rangle - \langle \mathbf{x}(t), \mathbf{y}(t+h)\rangle + \langle \mathbf{x}(t), \mathbf{y}(t+h)\rangle - \langle \mathbf{x}(t), \mathbf{y}(t)\rangle}{h} \\
&= \lim_{h\to 0} \frac{\langle \mathbf{x}(t+h) - \mathbf{x}(t), \mathbf{y}(t+h)\rangle + \langle \mathbf{x}(t), \mathbf{y}(t+h) - \mathbf{y}(t)\rangle}{h} \\
&= \lim_{h\to 0}\langle \frac{\mathbf{x}(t+h) - \mathbf{x}(t)}{h}, \mathbf{y}(t+h)\rangle + \lim_{h\to 0}\langle \mathbf{x}(t), \frac{\mathbf{y}(t+h) - \mathbf{y}(t)}{h}\rangle \\
&= \langle \mathbf{x}'(t), \mathbf{y}(t)\rangle + \langle \mathbf{x}(t), \mathbf{y}'(t)\rangle
\end{aligned}
$$

The last step is due to the following lemma (And you don't have to prove this in test/exam):

Lemma 6.1. *If $f, g : \mathbb{R} \to \mathbb{R}^n$ are bounded continuous functions. Then for all $a \in \mathbb{R}$, we have $\lim\limits_{x\to a} < f(x), g(x) >=< f(a), g(a) >$.*

This can be proved by Cauchy Schwartz inequality and understanding on continuous functions.

ii (1) *Note that as $\{\mathbf{e_i}\}_{i=1}^n$ is an basis of V, there exists $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$ such that*

$$\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{e_i}.$$

Then by taking inner product on both sides with $\mathbf{e_k}$ for $1 \le i \le n$, we have

$$
\begin{aligned}
&< \mathbf{x}, \mathbf{e_k} > =< \sum_{i=1}^n \alpha_i \mathbf{e_i}, \mathbf{e_k} > \\
&< \mathbf{x}, \mathbf{e_k} > = \sum_{i=1}^n \alpha_i < \mathbf{e_i}, \mathbf{e_k} > \\
&< \mathbf{x}, \mathbf{e_k} > = \alpha_k < \mathbf{e_k}, \mathbf{e_k} > + \sum_{i=1, i\neq k}^n \alpha_i < \mathbf{e_i}, \mathbf{e_k} > \\
&< \mathbf{x}, \mathbf{e_k} > = \alpha_k(1) + \sum_{i=1, i\neq k}^n (0) \ \textit{(By definition of orthonormal basis)} \\
&< \mathbf{x}, \mathbf{e_k} > = \alpha_k
\end{aligned}
$$

Hence we have $\mathbf{x} = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{e_i} \rangle \mathbf{e_i}$ *Also, note that*

$$
\begin{aligned}
\|\mathbf{x}\|^2 &=< \mathbf{x}, \mathbf{x} > \\
&=< \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{e_i} \rangle \mathbf{e_i}, \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{e_i} \rangle \mathbf{e_i} > \\
&= \sum_{i=1}^{n} < \langle \mathbf{x}, \mathbf{e_i} \rangle \mathbf{e_i}, \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{e_i} \rangle \mathbf{e_i} > \\
&= \sum_{i=1}^{n} \|\langle \mathbf{x}, \mathbf{e_i} \rangle\|^2 < \mathbf{e_i}, \mathbf{e_i} > \quad \textit{(As } \{\mathbf{e_i}\}_{i=1}^{n} \textit{ is an orthonormal basis.} \\
&= \sum_{i=1}^{m} \|\langle \mathbf{x}, \mathbf{e_i} \rangle\|^2 + \sum_{i=m+1}^{n} \|\langle \mathbf{x}, \mathbf{e_i} \rangle\|^2 \\
&\geq \sum_{i=1}^{m} \|\langle \mathbf{x}, \mathbf{e_i} \rangle\|^2 \quad \textit{(As } \|\langle \mathbf{x}, \mathbf{e_i} \rangle\|^2 \geq 0 \textit{ for all } i = 1, 2, ..., n.\textit{)}
\end{aligned}
$$

(2) *The statement is false.*

(Recall from tutorial notes 2, this is true only when the given basis is orthonormal).

Counter-example“ $\mathbf{v_1} = (2, 0), \mathbf{v_2} = (0, 1)$.

Then note that $(1, 0) = \frac{1}{2}(2, 0) + 0(0, 1) = \frac{1}{2}\mathbf{v_1}$.

However, note that $< (1, 0), \mathbf{v_1} >=< (1, 0), (2, 0) >= 2 \neq \frac{1}{2}$.

iii *Refer to solution of problem set 2 for more details.*