2024 TDG Group 2 Tutorial 9 (Final Revision)
26 Aug, 2024
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1 Syllabus Overview

Hope you enjoyed the course! Before the final, I hope you will be able to:

- Ch1 - Linear Algebra
  - Compute the basic operations: Addition, multiplication, dot and cross product relating to matrices and vectors.
  - Define and compute eigenvalues and eigenvectors given a 2×2 matrix.
  - Understand the geometrical meaning of various linear algebra terms: orthogonal/normal, linear independence, span, and basis, in the context of $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Ch2 - for *plane curves* $(a, b) \to \mathbb{R}^2$ and *space curves* $(a, b) \to \mathbb{R}^3$:
  - Define and check whether a curve $\mathbf{r}(t)$ is regular/p.b.a.l. (parametrized by arc length).
  - Define and compute the length of a curve.
  - Compute a p.b.a.l parametrization given a (regular) curve, recall and apply the uniqueness theorem on p.b.a.l. curves.
  - Define and compute $\mathbf{T}, \mathbf{N}, \kappa$ for a regular **plane curve**.
  - Define and compute (everything in the Fernet Formula) $\mathbf{T}, \mathbf{N}, \mathbf{B}, \kappa, \tau$ for a **space curve**.
  - Apply the Fernet formula to prove easy results (e.g. show some curve is contained in a plane), sometimes by differentiating them with respect to a variable.
- Ch3 - Given surface S and parametrization $\mathbf{x}(u, v)$:
  - Check that a parametrization is regular.
  - Define and compute $I, II, K = \det(II)/\det(I)$ for a surface at any point $p \in S$.
  - Define and compute Mean curvature H and show a surface is a minimal surface ($H = 0$ for all points p).
  - Understand the definition of isometry and theorema egregium.
  - Compute $\iint_S K\, dA$, recall and apply the Gauss-Bonnet Theorem.

Note that most are Define and compute questions, sometimes we also ask you to show or proof something, those are harder.

2 Question Bank

Please do Q1, 2, 4, 7, 10 ASAP.

1. Are the following curves regular? Explain your answer.
   (a) $\mathbf{r} : (-1, 1) \to \mathbb{R}^2 :, \mathbf{r}(t) = (t^2, t^3)$.
   (b) $\mathbf{r} : (0, 2) \to \mathbb{R}^2 :, \mathbf{r}(t) = (t^2, t^3)$.
   (c) $\mathbf{r} : (-1, 1) \to \mathbb{R}^3 :, \mathbf{r}(t) = (t, t^2 + 1, t - 1)$.

2. Find the curvature and torsion of $\mathbf{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$, and the vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$.

3. **(Challenging.)** Let $\mathbf{r} : (a, b) \to \mathbb{R}^3$ be a regular space curve parametrized by arc length. Assume that the curvature and torsion are nowhere zero, i.e. $\kappa(s) \neq 0, \tau(s) \neq 0$, for all $s \in (a, b)$, and there exists a unit vector $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{r}'(s)$ makes a constant angle with $\mathbf{v}$ for all $s \in (a, b)$.
   (a) Show that $\mathbf{n}(s) \cdot \mathbf{v} = 0$ for all $s \in (a, b)$.
   (b) Show that the angle between $\mathbf{v}$ and binormal vector $\mathbf{b}$ is constant. (Hint: You may write $\mathbf{v}$ as a linear combination of $\mathbf{t}, \mathbf{n}, \mathbf{b}$.)
   (c) Show that for some $c \in \mathbb{R}$, we have $\frac{\kappa(s)}{\tau(s)} = c$ for every $s \in (a, b)$.

4. (2013, Exam Q2) By considering the function
$$\boldsymbol{f}(s) := \boldsymbol{\alpha}(s) + \frac{1}{\kappa(s)}\mathbf{N}(s),$$
show that if $\boldsymbol{\alpha}(s)$ is a unit speed curve with constant curvature $\kappa > 0$ and torsion $\tau(s) \equiv 0$, then $\boldsymbol{\alpha}$ is part of a circle with radius $\frac{1}{\kappa}$. You may use the fact that a curve with positive curvature and zero torsion is a plane curve.

5. Let S be the surface parametrized by
$$\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3, \quad \mathbf{x}(u, v) = (u + v, u - v, uv).$$
With respect to this parametrization:
   (a) Show that $\mathbf{x}$ is regular.
   (b) Find the first and second fundamental form of S.
   (c) Find the mean and Gaussian curvatures of S.

6. *(In lecture.)* Let S be the surface parametrized by
$$\mathbf{x} : (-\pi, \pi) \times \mathbb{R} \to \mathbb{R}^3, \quad \mathbf{x}(u, v) = (\cos(u)\cosh(v), \sin(u)\cosh(v), v).$$
With respect to this parametrization,
   (a) Define and compute the Gauss map of this surface.
   (b) Show that S is a minimal surface.

(You may use any of the identities $\frac{d}{dt}\sinh(t) = \cosh(t)$, $\frac{d}{dt}\cosh(t) = \sinh(t)$, and $\cosh^2(t) - \sinh^2(t) = 1$.)

7. Let S be the surface parametrized by

$$\mathbf{x}(u, v) = (v, vu, vu^2)$$

for positive real numbers u, v.

(a) Show that $\mathbf{x}$ is regular.

(b) Find the principal curvature, principal vectors, mean curvature, gaussian curvature at $p = (1, 1, 1)$ in S.

8. (2017 Final Exam, Q4).

(a) Let $\mathbf{x}(u, v) = (f_1(u, v), f_2(u, v), f_3(u, v))$ be a regular surface such that the coefficients of the First Fundamental Form satisfy the equations $E = G, E, G$ are always non-zero, and $F = 0$, where f_1, f_2 and f_3 are smooth functions. Show that

i. $\langle \mathbf{x}_{uu} + \mathbf{x}_{vv}, \mathbf{n} \rangle = -2EH$, where H and $\mathbf{n}$ are mean curvature and unit surface normal vector of $\mathbf{x}$ respectively.

ii. If the functions f_1, f_2, f_3 satisfy

$$\frac{\partial^2 f_i}{\partial u^2} + \frac{\partial^2 f_i}{\partial v^2} = 0 \text{ for } i = 1, 2, 3,$$

then $\mathbf{x}(u, v)$ is a minimal surface.

(b) Further let $\mathbf{y}(u, v) = (g_1(u, v), g_2(u, v), g_3(u, v))$ be a regular surface, where g_1, g_2 and g_3 are smooth functions. Let E^y, F^y, G^y be the First Fundamental Form of y. Suppose $E^y = G^y, E^y, G^y$ are always non-zero and $F^y = 0$, both $\mathbf{x}, \mathbf{y}$ are minimal surfaces and the coordinates of $\mathbf{x}, \mathbf{y}$ satisfy

$$\frac{\partial f_i}{\partial u} = \frac{\partial g_i}{\partial v} \text{ and } \frac{\partial f_i}{\partial v} = -\frac{\partial g_i}{\partial u} \text{ for } i = 1, 2, 3.$$

Let $\mathbf{z}^t(u, v) = (\cos(t))\mathbf{x}(u, v) + (\sin(t))\mathbf{y}(u, v)$, where $t \in \mathbb{R}$. Using (a)(ii), or otherwise, show that $\mathbf{z}^t$ also is a minimal surface.

9. Let S be the surface parametrized by , $\mathbf{x}(u, v) = (\cos(u), \sin(u), v)$ for all $u, v \in \mathbb{R}$. Find all unit-speed smooth curves

$$\mathbf{r} : \mathbb{R} \to S \subset \mathbb{R}^3, \quad \mathbf{r}(t) = \mathbf{x}(u(t), v(t)),$$

so that for all $t, \mathbf{r}'(t)$ is a principal vector of S at $\mathbf{r}(t)$.

10. (2018,Final Q4) Let M be a smooth closed surface with Gauss curvature K.

(a) If $K < 0$ entirely on M, what is the minimum genus of M?

(b) If $K < -4$ entirely on M and the Euler characteristic of M is -6, show that the surface area of M is not more than 3π.