2024 TDG Group 2 Tutorial 8 (Curvatures of Surfaces)
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1 Motivation

Given a curve $\mathbf{r}(t)$, one part of the Fernet formula reads:

$$\frac{d}{ds}\mathbf{T}(s) = \kappa\mathbf{N}(s)$$

So the curvature κ measures the *change of the tangent line*. The tangent line always "changes" along the normal direction, so a real number suffices to capture it.

Now we move up one dimension, the "curvature" we want captures the change of the tangent *plane*. Now the tangent plane moves in *two* direction (say, the $\mathbf{x}_u$ and $\mathbf{x}_v$ direction), how can we capture the "change"?

2 Linear Algebra Detour

On this note, all matrices will be 2x2 unless otherwise specified. Let's review some key notions from Ch1.

2.1 Eigenvalues and Eigenvectors

Recall that given a (2x2) matrix M, one can do the following procedure to find the eigenvalue and eigenvector:

- Solve for λ in $\det(M - \lambda\mathbf{I}) = 0$. M is 2x2 so the above equation is a quadratic. Those are your **eigenvalues**. $\mathbf{I}$ is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, not the first fundamental form.
- Substitute your 2 λs (now known) in

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

  and solve. You should get a relation like $5x = 3y$. Then your "eigenspace" is $\text{span}(3, 5)$, your **eigenvector** is (anything that is a multiple of) $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Example 1. Find the eigenvalue and eigenvector of $M = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$.

Ans. Note that $\det(M - \lambda I) = (4 - \lambda)(1 - \lambda) + 2 = 0$, so $\lambda = 2$ or 3. For $\lambda = 2$, solving $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$ gives $x - 2y = 0$, so an eigenvector to $\lambda = 2$ is $(2, 1)$ (or any multiples of it), similarly for $\lambda = 3$ gives $x - y = 0$, so a eigenvector associated to $\lambda = 3$ is $(1, 1)$.

Actually, the polynomial $\det(M - \lambda \mathbf{I})$ gives you more information than you might expect!

Exercise 1 (Warming Up.)**.** In all the following exercises, please do the **bolded** questions first.

1. Find the eigenvalue and eigenvector of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

2. Given a diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, what are its eigenvalue and eigenvectors when $a \neq b$? How about when $a = b$?

3. Given a 2x2 matrix M, show that the polynomial $p(x) := \det(M - x\mathbf{I})$ is actually

$$p(x) = x^2 - \operatorname{tr}(M)x + \det M$$

, recall that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\operatorname{tr}(M) = a + d$ and $\det(M) = ad - bc$.

Remember your DSE Core Maths, since the roots of $p(x)$ *are the eigenvalues*, hence

The sum of eigenvalues is the trace, the product of eigenvalues is the determinant!

3 The computation

Now, given a surface S with parametrization $\mathbf{x}(u, v)$ and a point $p \in S$, we know:

- First fundamental form
$$\mathrm{I}(u, v) = \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle & \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle & \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix} = \begin{pmatrix} E(u, v) & F(u, v) \\ F(u, v) & G(u, v) \end{pmatrix}$$
- Second fundamental form
$$\mathrm{II}(u, v) = \begin{pmatrix} \langle \mathbf{x}_{uu}, \mathbf{n} \rangle & \langle \mathbf{x}_{uv}, \mathbf{n} \rangle \\ \langle \mathbf{x}_{vu}, \mathbf{n} \rangle & \langle \mathbf{x}_{vv}, \mathbf{n} \rangle \end{pmatrix} = \begin{pmatrix} e(u, v) & f(u, v) \\ f(u, v) & g(u, v) \end{pmatrix}$$
- Gauss Curvature
$$K(u, v) = \frac{\det \mathrm{II}(u, v)}{\det \mathrm{I}(u, v)}$$

Key points:

- For each $p \in S$, we can write $p = \mathbf{x}(u_0, v_0)$, then the fundamental forms gives a matrix $I(u_0, v_0), II(u_0, v_0)$ at each point p, and the Gaussian curvature gives a real number at each point p.
- Doesn't matter the argument is u, v or s, θ, the calculation is exactly the same!

Recall the Gauss map $\mathcal{G}$ assign each point p to its normal vector $\mathbf{n}_p$. Through a process d we obtain the differential of this map. That is, for each point p on S we have the matrix:

$$d\mathbf{n}_p = -(\mathrm{II})\left(\mathrm{I}^{-1}\right) = -\frac{1}{EG - F^2} \begin{pmatrix} eG - fF & fE - eF \\ fG - gF & gE - fF \end{pmatrix}. \tag{1}$$

This has another name called **the shape operator**.

3.1 Curvature from the Shape Operator

We will be investigating $d\mathbf{n}_p$ today. In (1), the matrix is a linear map $T_pS \to T_pS$, so it is with respect to the basis $\mathbf{x}_u$ as $(1,0)$ and $\mathbf{x}_v$ as $(0,1)$ now recall:

Definition 1. The **principal curvature** κ_1, κ_2 and **principal direction** $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ are the *negative* of eigenvalues of $d\mathbf{n}_p$ and (any) corresponding eigenvector of $d\mathbf{n}_p$ respectively.

Definition 2. The **Gaussian Curvature** and **Mean curvature** is $K = \det(d\mathbf{n}_p)$ and $H = -\dfrac{1}{2}\operatorname{tr}(d\mathbf{n}_p)$. A surface S is called a **minimal surface** if $H = 0$ at every point $p \in S$.

By the above exercise, if we have $d\mathbf{n}_p$ in matrix, then writing down $\det(d\mathbf{n}_p - x\mathbf{I})$ would give us K and H! Or...

Theorem 1.

$$K = \kappa_1\kappa_2 = \frac{eg - f^2}{EG - F^2}$$

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\left(\frac{gE - 2fF + eG}{EG - F^2}\right).$$

Proof. Directly computing $\det(d\mathbf{n}_p)$ and $-\dfrac{1}{2}\operatorname{tr}(d\mathbf{n}_p)$ as in (1). □

Example 2. Let S be the surface parametrized by

$$\mathbf{x}(u,v) = ((2+\sin u)\cos(v), (2+\sin u)\sin v, \sin u)\,. \tag{2}$$

where $0 < u, v < 2\pi$. With respect to this parametrization:

(a) Show that $\mathbf{x}$ is regular.

(b) Find the first and second fundamental form of S.

(c) Find the principal curvature, mean and Gaussian curvatures of S.

Ans. (a) The computation goes:

$$\begin{aligned}
\mathbf{x}_u &= (\cos(u)\cos(v), \cos(u)\sin(v), -\sin(u)) \\
\mathbf{x}_v &= (-(\sin(u)+2)\sin(v), (\sin(u)+2)\cos(v), 0) \\
\mathbf{x}_u \times \mathbf{x}_v &= (\sin u + 2)(\cos(v)\sin(u), \sin(u)\sin(v), \cos(u))
\end{aligned}$$

Be careful of the regularity argument here: Assume for the contrary, that $\mathbf{x}$ is not regular, so there might be some u, v with $\mathbf{x}_u \times \mathbf{x}_v = \mathbf{0}$. Now, $\sin u + 2 \neq 0$ for any u so in order for $\mathbf{x}_u \times \mathbf{x}_v = \mathbf{0}$, we must have *all three* entries in the right-hand side bracket to be 0. $\sin(u)\sin(v) = 0$ So either $\sin(u) = 0$ or $\sin(v) = 0$, they are both in range of $(0, 2\pi)$ so either $u = \pi$ or $v = \pi$. If $u = \pi$, then the third entry $\cos u$ won't be zero. If $v = \pi$, then the first entry $\cos(v)\sin(u)$ becomes $-\sin(u)$, and is zero only if $\sin(u) = 0$, so $u = \pi$ and the previous arguments applies.

In any case, it is *impossible* that $\mathbf{x}_u \times \mathbf{x}_v = \mathbf{0}$, so $\mathbf{x}$ is a regular parametrization.

(b) More computation:

$$
\begin{aligned}
|\mathbf{x}_u \times \mathbf{x}_v| &= \sin(u) + 2 \\
\mathbf{x}_{uu} &= (-\cos(v)\sin(u), -\sin(u)\sin(v), -\cos(u)) \\
\mathbf{x}_{uv} &= (-\cos(u)\sin(v), \cos(u)\cos(v), 0) \\
\mathbf{x}_{vv} &= (-(\sin(u)+2)\cos(v), -(\sin(u)+2)\sin(v), 0) \\
\begin{pmatrix} E & F \\ F & G \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & (\sin u + 2)^2 \end{pmatrix} \\
\begin{pmatrix} e & f \\ f & g \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & -\sin u(\sin u + 2) \end{pmatrix}
\end{aligned}
$$

(c) Note that I and II are diagonal matrices, so just invert and multiply the matrix element-wise, hence we have

$$
\begin{aligned}
d\mathbf{n}_p &= \begin{pmatrix} -1 & 0 \\ 0 & \dfrac{-\sin u}{\sin u + 2} \end{pmatrix} \\
K &= \frac{\sin(u)}{\sin(u) + 2} \\
H \text{ Mean curvature: } & \frac{\cos(u)^2 - 3\sin(u) - 3}{\sin(u)^2 + 4\sin(u) + 4}
\end{aligned}
$$

(and the principal curvatures are diagonal entries of $d\mathbf{n}_p$.)

Exercise 2. Given the sphere of radius $R > 0$ parametrized by

$$\mathbf{x}(\phi, \theta) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)$$

where $0 < \phi < \pi, 0 < \theta < 2\pi$, find the principal curvature, gaussian curvature and mean curvature.

Exercise 3. Ch.3 Exercise (p. 143): 14c, 15, **19**, 21

3.2 Total Curvature and Euler Characteristic

Let's investigate the two sides of the Gauss-Bonnet equation.

Theorem 2. *If S is a simple closed regular surface then*

$$\iint_S K\, dA = 2\pi\chi(S).$$

Definition 3. Let $\mathbf{x}(u, v)$, $\mathbf{x} : D \to \mathbb{R}^3$ be a parametrization of surface S and let $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ be the corresponding first fundamental form. For a (smooth) function $f : S \to \mathbb{R}$, we define the **surface integral of f on S** to be

$$\iint_S f dA = \int_D f \circ \mathbf{x} \left(EG - F^2\right)^{1/2} dudv = \iint_D f(\mathbf{x}(u, v)) \cdot \sqrt{\det \mathrm{I}(u, v)} dudv.$$

The middle $\cdot$ is the multiplication of two functions, to be clear.

Theorem 3. *Given a surface S, then*

$$\text{Area}(S) = \iint_S dA = \iint_S 1dA.$$

So $\iint_S K\,dA$ is just the $f(\mathbf{x}(u,v))$ part replaced with K (expressed in u, v). Intuitively, you can think of this as "adding all the K at each point of the surface". For this reason, $\iint_S K\,dA$ is also called the **total curvature** of S.

Now a note on the Euler characteristic $\chi(S)$, it can be found by triangulation. There are three important points:

1. Two triangulation (maybe with different V, E, F) gives the same Euler Characteristic on the same surface S, and
2. The Euler characteristic is unchanged under *homeomorphism* - that is, stretching and compressing the surface.
3. (Theorem 3.6.4) A simple closed regular surface S which is (homeomorphic to) a "sphere with g holes" has Euler Characteristic $\chi(S) = 2 - 2g$.

Example 3. Verify the Gauss-bonnet equation for the Torus parametrized by $\mathbf{x}$ in (2).

Ans. Recall that a torus has euler characteristic $2-2(1) = 0$. Let us verify $\iint_S K\,dA = 0$. This is not hard. Remember the bound is $0 < u, v < 2\pi$:

$$\iint_S K\,dA = \int_0^{2\pi}\int_0^{2\pi} K(u,v)\,du\,dv = \int_0^{2\pi}\int_0^{2\pi} \frac{\sin(u)}{\sin(u)+2}\cdot\sqrt{(\sin u + 2)^2}\,du\,dv = \int_0^{2\pi}\int_0^{2\pi} \sin u\,du\,dv = 0$$

Exercise 4. **1.** Verify the sphere of radius R satisfies the Gauss-Bonnet Equation.

2. Ch3 Excercise, **Q18a)**, 22 (p.146)
3. If S is a simple closed regular surface that is homeomorphic to a sphere, show that there is a point p of S at which $K(p) > 0$.

4 Non-examinable Materials

4.1 Billinear forms

When investigating a matrix, one might think about the change of basis... you might heard that this is the act of expressing it as a linear combination other than the usual $(1,0), (0,1)$. Recall that this is the same as giving you an invertible matrix P so it turns the matrix M to $P^{-1}MP$.

Definition 4. A matrix M is **similar** to a matrix N if there exists a matrix P such that $N = P^{-1}MP$.

Do let me know if you want to listen more...