SAYT1134 Towards Differential Geometry
Group 3 Tutorial 2

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1 Cross Product

1. Find $\mathbf{u} \times \mathbf{v}$, where
   (a) $\mathbf{u} = (1, 0, 2), \mathbf{v} = (2, 3, 4)$
   (b) $\mathbf{u} = (-8, -2, -4), \mathbf{v} = (2, 2, 1)$
2. Find the area of triangle determined by the points P, Q, and R.
   (a) $P(1, -1, 2), Q(2, 0, -1), R(0, 2, 1)$
   (b) $P(1, 1, 1), Q(2, 1, 3), R(3, -1, 1)$
   (c) $P(2, -2, 1), Q(3, -1, 2), R(3, -1, 1)$
3. Find a unit vector perpendicular to plane PQR, where P, Q, and R are points in Question 2.
4. For the following $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, verify that

$$\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \times \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{w} \times \mathbf{u} \rangle.$$

   (a) $\mathbf{u} = (1, -1, 1), \mathbf{v} = (2, 1, -2).\mathbf{w} = (-1, 2, -1)$
   (b) $\mathbf{u} = (2, 1, 0), \mathbf{v} = (2, -1, 1), \mathbf{w} = (1, 0, 2)$
   (c) $\mathbf{u} = (1, 1, -2), \mathbf{v} = (-1, 0, -1), \mathbf{w} = (2, 4, -2)$

2 More about Integration

1. Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx$
2. Evaluate $\displaystyle\int \tan^4 x \, dx$

3. Evaluate $\int \sec^3 x\, dx$

4. Evaluate $\int \sqrt{1-9t^2}\, dt$

5. Evaluate $\int \frac{1}{x^2\sqrt{x^2-1}}\, dx$, $x > 1$

6. Evaluate $\int \sqrt{\frac{4-x}{x}}\, dx$ (Hint: Let $x = u^?$)

3 Curves

1. Verify if the function $\mathbf{r}(t) = (t, t^2)$, $t \in \mathbb{R}$ is a regular parametrized curve.

2. Verify if the function $\mathbf{r}(t) = (t^3 - 4t, t^2 - 4)$, $t \in \mathbb{R}$ is a regular parametrized curve. Also, show that it is not injective.

3. What is the parametrization of the line through $(1, 0, 9)$ and $(5, 4, 2)$?

4. Find a parametrized curve $\mathbf{r}(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\mathbf{r}(t)$ runs clockwise around the circle with $\mathbf{r}(0) = (0, 1)$.

5. Let $\mathbf{r}(t)$ be a parametrized curve which does not pass through the origin. If $\mathbf{r}(t_0)$ is the point of the trace of $\mathbf{r}$ closest to the origin and $\mathbf{r}$'$(t_0) \neq \mathbf{0}$, show that the position vector $\mathbf{r}(t_0)$ is orthogonal to $\mathbf{r}'(t_0)$.

6. Let $\mathbf{r}\colon I \to \mathbb{R}^3$ be a parametrized curve and let $\mathbf{v} \in \mathbb{R}^3$ be a fixed vector. Assume that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{v}$ for all $t \in I$ and that $\mathbf{r}(0)$ is also orthogonal to $\mathbf{v}$. Prove that $\mathbf{r}(t)$ is orthogonal to $\mathbf{v}$ for all $t \in I$.

7. Let $\mathbf{r}\colon I \to \mathbb{R}^3$ be a parametrized curve, with $\mathbf{r}'(t) \neq 0$ for all $t \in I$. Show that $|\mathbf{r}(t)|$ is a nonzero constant if and only if $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$ for all $t \in I$.

4 Arc length

1. Show that $\mathbf{r}(s) = (a\cos(\frac{s}{c}), a\sin(\frac{s}{c}), b\frac{s}{c})$ is parametrized by arc length, where $c^2 = a^2 + b^2$.

2. Consider $\mathbf{r}(t) = (\cos(5t), \sin(5t), 12t)$. What is the speed of $\mathbf{r}$? Find the arc-length parametrization $\mathbf{r}(s)$.

3. Compute the arc-length of logarithmic spiral given by $\mathbf{r}(t) = (ae^{bt}\cos t, ae^{bt}\sin t, 0)$