3.6 Gauss-Bonnet theorem

Theorem 3.6.6 (Gauss-Bonnet theorem). *Let S be a simple closed regular surface in $\mathbb{R}^3$. Then*

$$\iint_S K\,dA = 2\pi\chi(S)$$

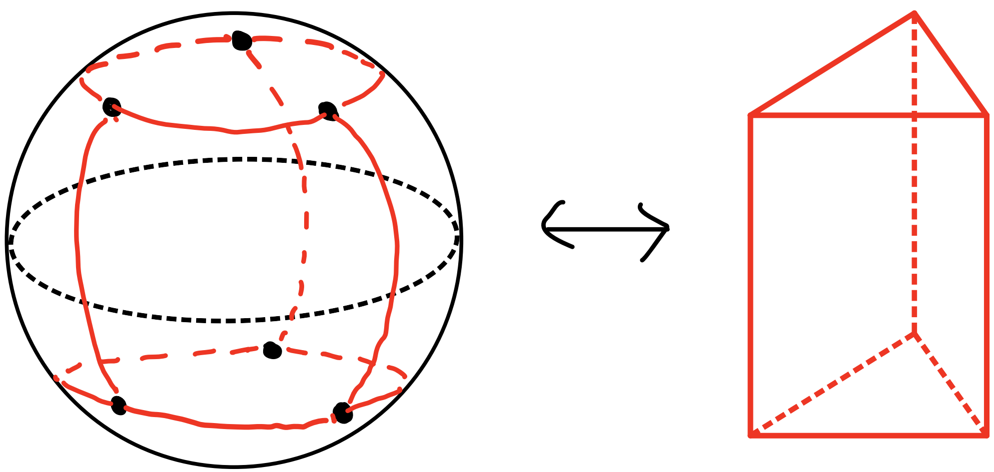
The Theorem relates local geometry (K) with global shape $\chi(S)$

Definition 3.6.1 (Euler characteristic). *The* **Euler characteristic** *of a closed surface S is*

$$\chi(S) = v - e + f$$

where v, e and f are the number of vertices, edges and faces of a polyhedron modeled on S.

eg



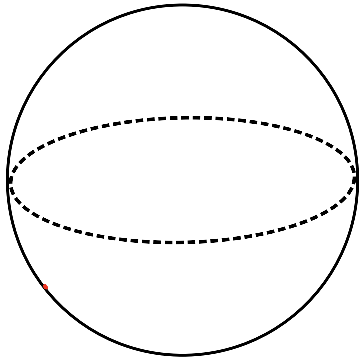
$v = 6$

$e = 9$

$f = 5$

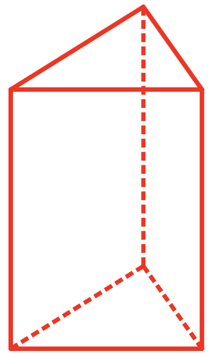
$\chi(S^2) = v - e + f = 2$

Same $\chi(S)$ for any polyhedron modeled on S?

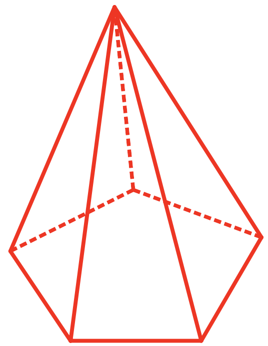


$$\chi(S^2) = v - e + f = 2$$

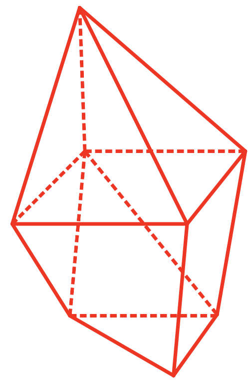
$\longleftrightarrow$



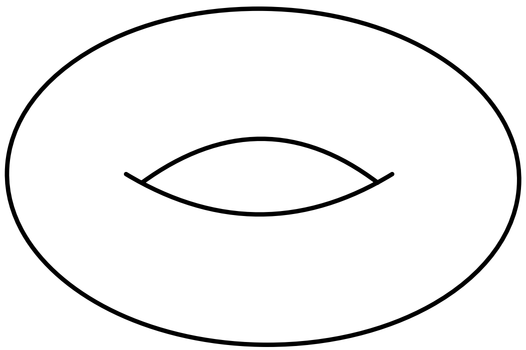
$v = 6$
$e = 9$
$f = 5$



$v = 6$
$e = 10$
$f = 6$



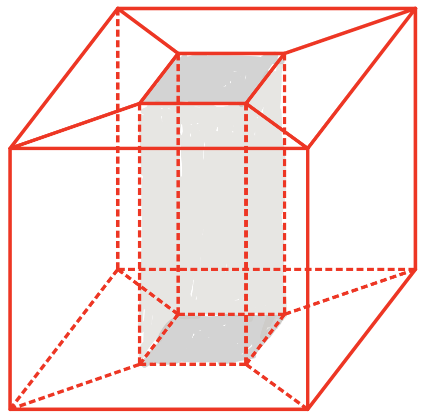
$v = 8$
$e = 16$
$f = 10$



T

$$\chi(T) = v - e + f = 0$$

$\longleftrightarrow$



$v = 16$
$e = 32$
$f = 16$

Theorem 3.6.2 (Area of polygon on unit sphere). *Let α, β, γ be the interior angles of a triangle, with edges being great circular arcs*[12]*, on the unit sphere and A be the area of the triangle. Then*

$$\alpha + \beta + \gamma = A + \pi. \qquad (1)$$

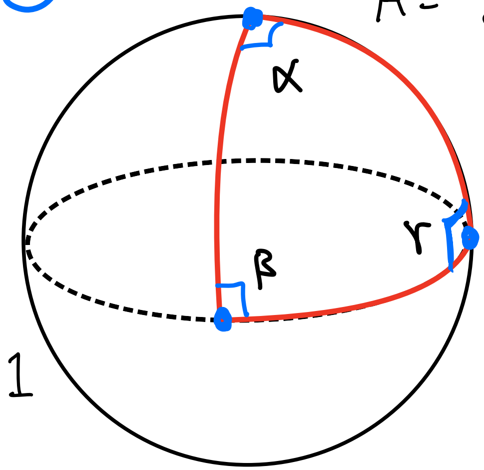
More generally, Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the interior angles of a polygon with n edges, which are great circular arcs, on the unit sphere and A be the area of the polygon. Then

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = A + (n-2)\pi. \qquad (2)$$

Angle sum of Δ may not be π if $K \neq 0$

eg (1)

$A = \frac{4\pi}{8} = \frac{\pi}{2}$



$r = 1$

triangle with 3 right angles

$\alpha + \beta + \gamma = \frac{3\pi}{2} = A + \pi$

Pf of (2) from (1)

Subdivide n-gon into n-2 triangles $\Delta_1, \Delta_2, \cdots \Delta_{n-2}$

For each triangle Δ_i

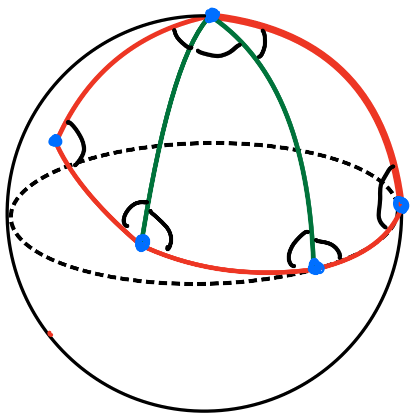
angle sum of $\Delta_i = A_i + \pi$

$\sum_{i=1}^{n-2}$ angle sum of $\Delta_i = \sum_{i=1}^{n-2}(A_i + \pi)$

angle sum of n-gon $= \sum_{i=1}^{n-2} A_i + (n-2)\pi$

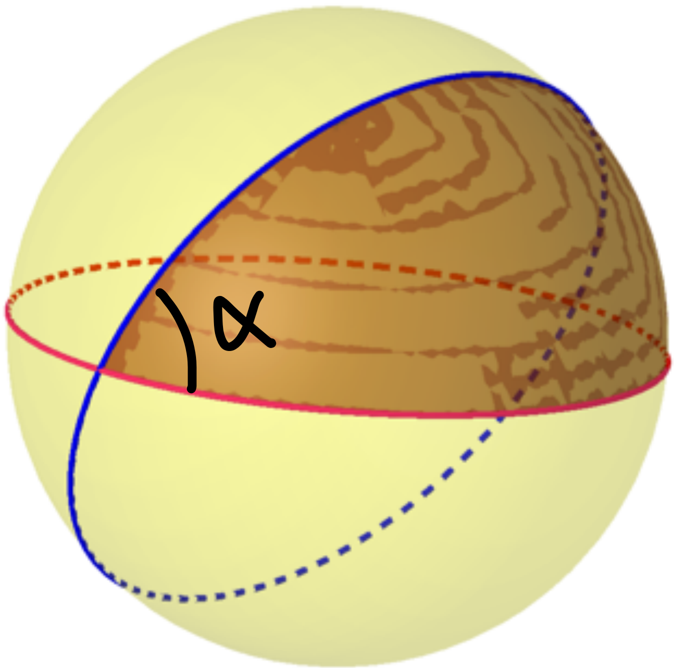
$= A + (n-2)\pi$

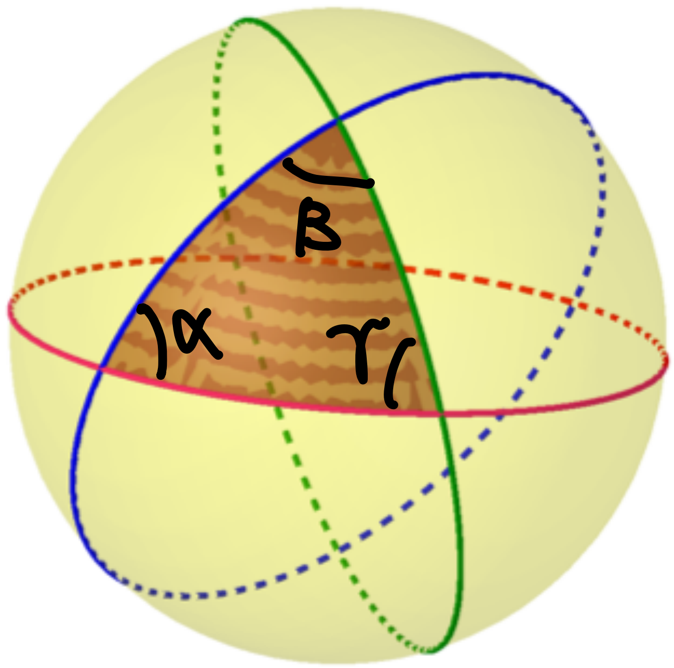
eg $n=5$:



subdivided into 3 triangles

Pf of 1 : $\alpha + \beta + \gamma = A + \pi$.





Area of biangle (interior angle α)

$$= 4\pi \cdot \frac{\alpha}{2\pi} = 4\alpha$$

Sum of Area of 6 biangles

$$2(2\alpha) + 2(2\beta) + 2(2\gamma) = 4\pi + 4A$$

$$\Rightarrow \alpha + \beta + \gamma = \pi + A$$

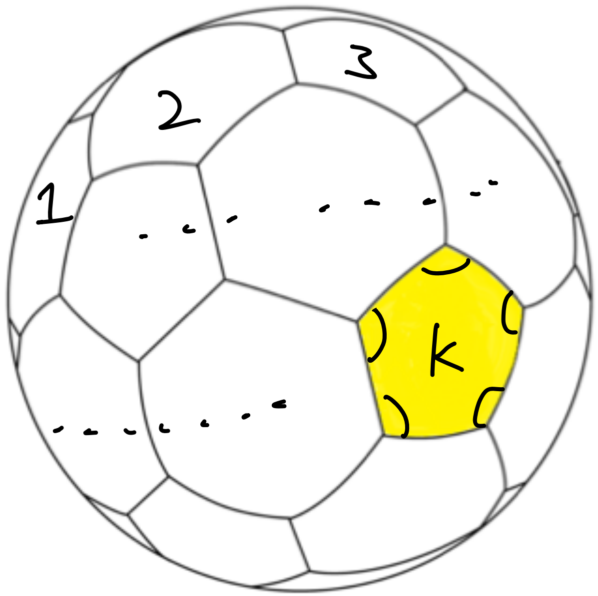
Theorem 3.6.3 (Euler characteristic of sphere). *A polyhedron which is modeled on a sphere has Euler characteristic $\chi = 2$.*

Proof. Consider a polyhedron modeled on the unit sphere. By deforming the edges, we may assume that the edges are great circular arcs on the unit sphere.

Label the faces $1, 2, 3, \cdots, f$

Let the k-th face have e_k edges

and angles $\alpha_{k1}, \alpha_{k2}, \cdots \alpha_{ke_k}$



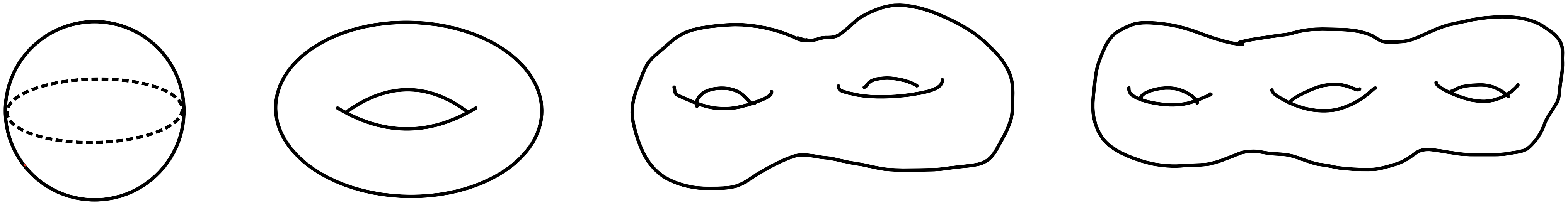
$$\sum_{i=1}^{e_k} \alpha_{ki} = (e_k - 2)\pi + A_k$$

$$\underbrace{\sum_{k=1}^{f}\sum_{i=1}^{e_k} \alpha_{ki}}_{2\pi v} = \underbrace{\sum_{k=1}^{f} e_k \pi}_{2\pi e} - \underbrace{2\sum_{k=1}^{f} \pi}_{2\pi f} + \underbrace{\sum_{k=1}^{f} A_k}_{4\pi}$$

$$\Rightarrow \quad v = e - f + 2$$

$$v - e + f = 2$$

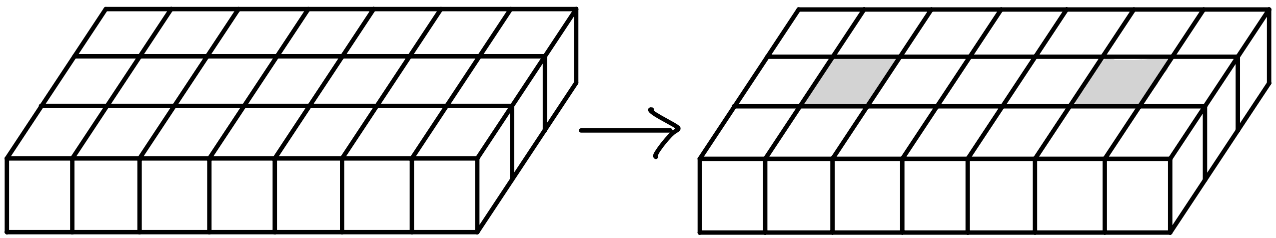
Genus of closed surfaces (Number of 'hole')



$g=0$	$g=1$	$g=2$	$g=3$
$\chi=2$	$\chi=0$	$\chi=-2$	$\chi=-4$

Theorem 3.6.4 (Euler characteristic of simple closed surface). *Let S be a simple closed surface of genus g. Then the Euler characteristic of S is*

$$\chi(S) = 2 - 2g.$$



$v - e + f = 2$

"Each hole created"

$$\left.\begin{array}{ll} g & +1 \\ v & \text{no change} \\ e & +4 \\ f & +2 \end{array}\right\} \Rightarrow \chi \ -2$$

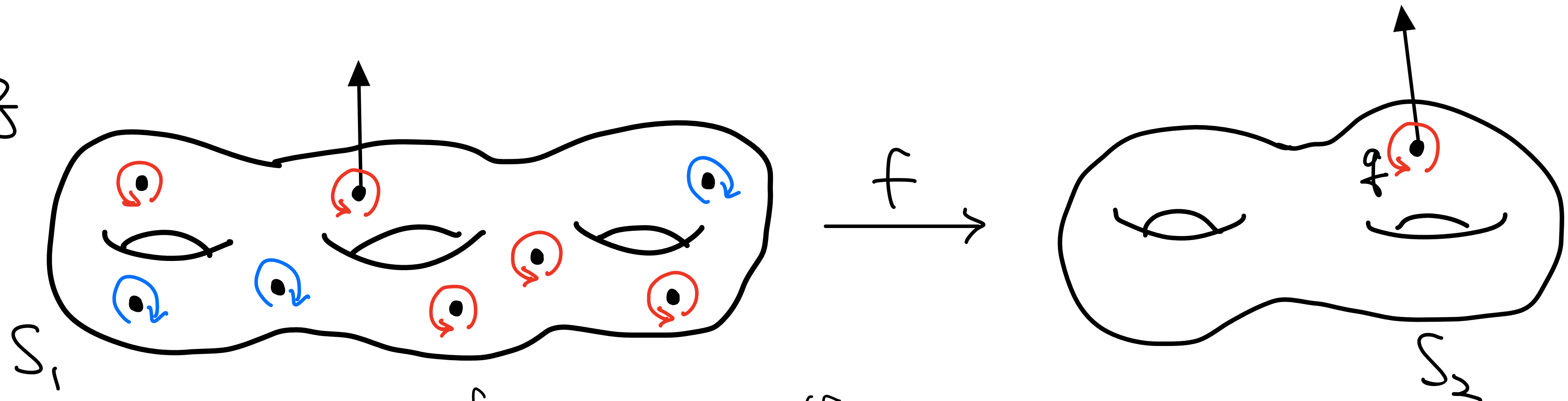
Degree of a map between surfaces

Let S_1 and S_2 be two simple closed surface in $\mathbb{R}^3$. Let $f : S_1 \to S_2$ be a continuous map from S_1 to S_2. For $q \in S_2$, we define the degree of f at q to be the integer

closed means bounded, no boundary

$$\boxed{\deg(f, q) = \begin{array}{l} \text{number of preimages of } q \text{ preserving orientation} \\ -\text{number of preimages of } q \text{ reversing orientation} \end{array}.}$$

eg



• preimage of q $\qquad \deg(f, q) = 5 - 3 = 2$

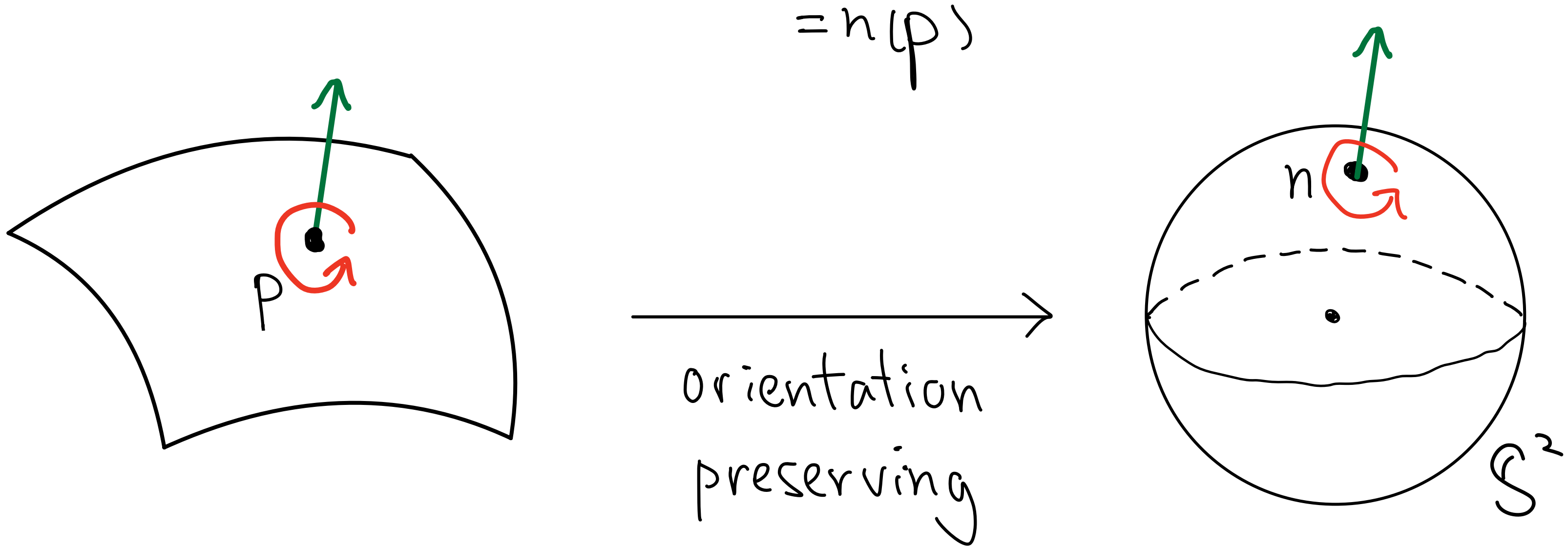
Rmk $\deg(f, q)$ is the same for any q with finite preimage

This number is called $\deg(f)$

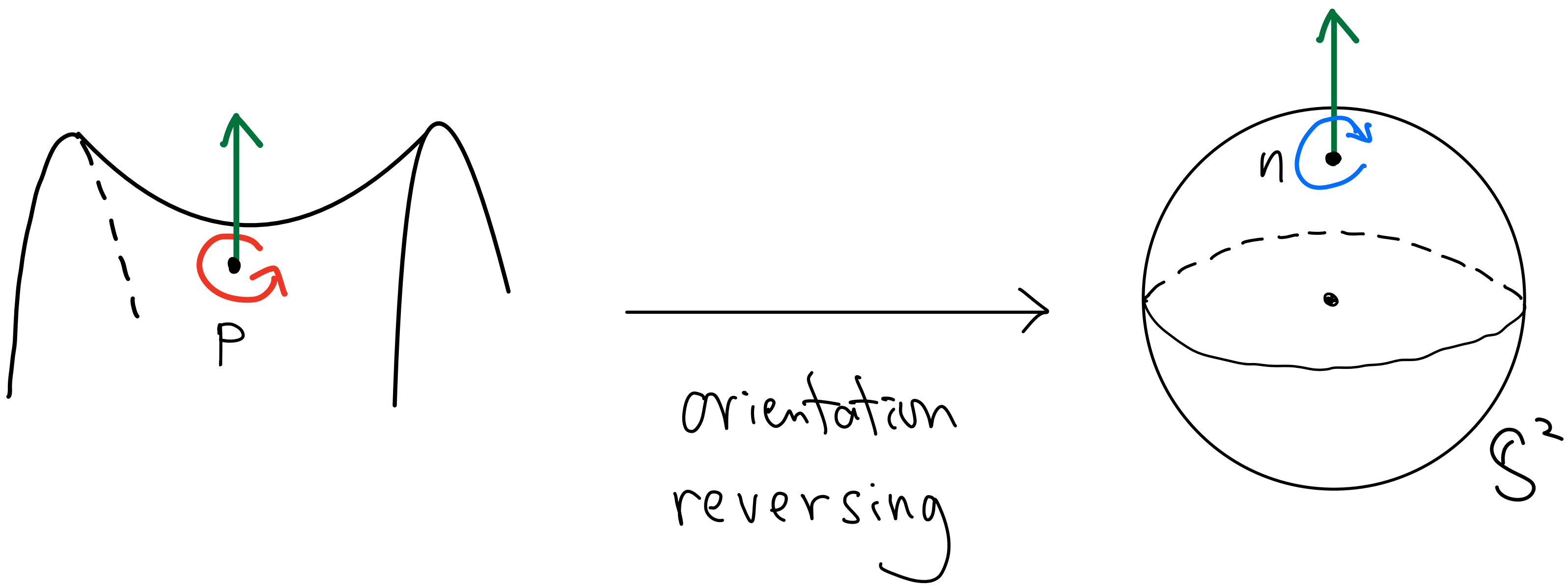
It counts the number of times "S_1 covers S_2 through f"

Degree of Gauss Map

$K(p) > 0$

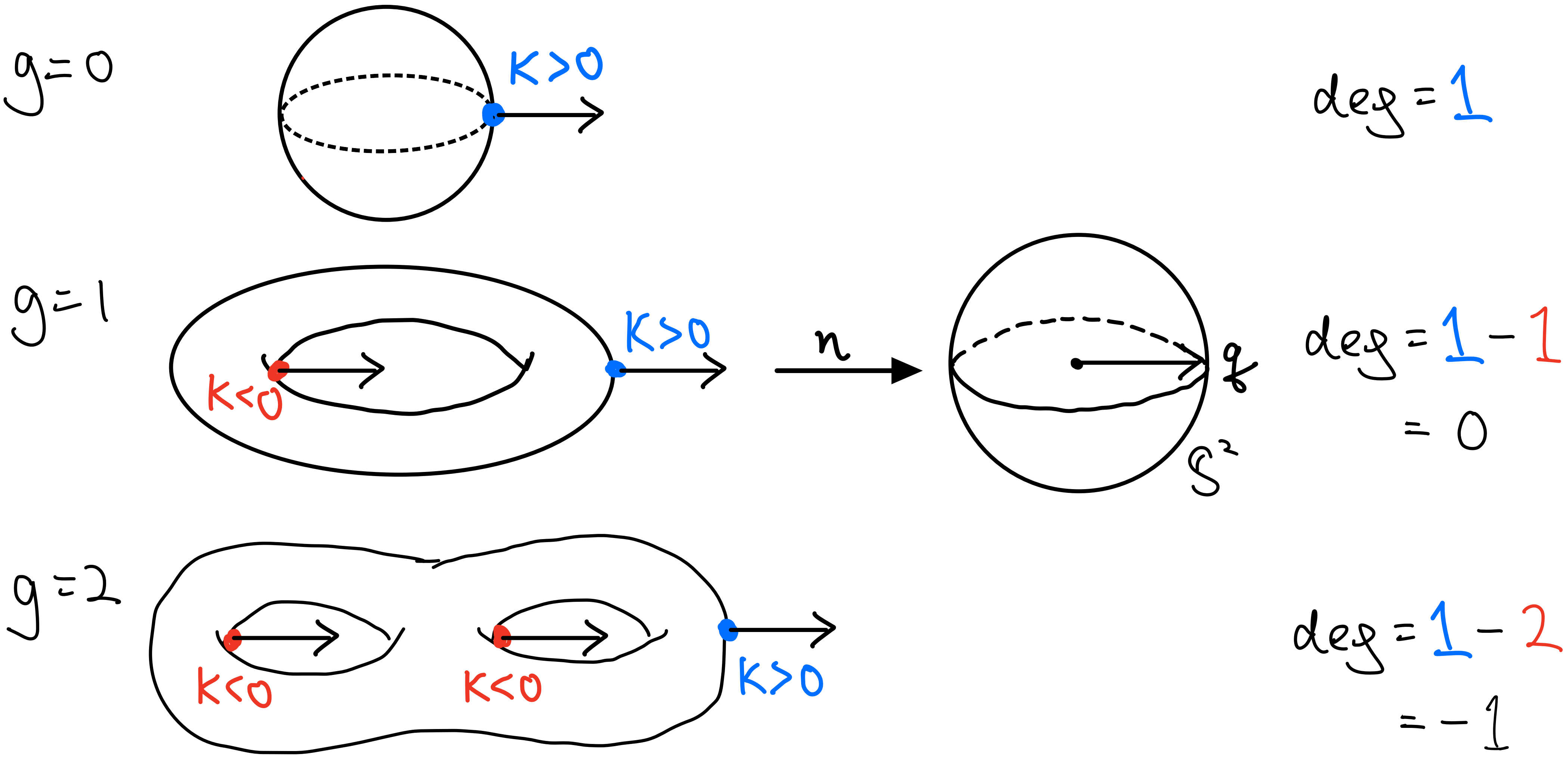


$K(p) < 0$



Theorem 3.6.5 (Degree of Gauss map of simple closed regular surface). *Let S be a simple closed surface of genus g. The the degree of Gauss map of S is*

$$\deg(\mathbf{n}) = 1 - g.$$



Theorem 3.6.6 (Gauss-Bonnet theorem). *Let S be a simple closed regular surface in $\mathbb{R}^3$. Then*

$$\iint_S K dA = 2\pi\chi(S)$$

where K is the Gaussian curvature, $\chi(S)$ is the Euler characteristic of S and $dA = \sqrt{\det(I)}dudv$ is the surface area element. In particular, if S is homeomorphic[13] *to the sphere S^2, then $\chi(S) = 2$ and*

$$\iint_S K dA = 4\pi.$$

<u>Pf</u>

$$\begin{aligned}\iint_S K dA &= \iint_S \frac{d\sigma}{dA} dA \\ &= \iint_S d\sigma \\ &= \deg(n) \iint_{S^2} d\sigma \\ &= (1-g)(4\pi) \\ &= 2\pi(2-2g) = 2\pi\chi(S)\end{aligned}$$