

3.3 Second fundamental form and Gaussian curvature

Example 3.3.8 (Catenoid). Consider the surface obtained by rotating the catenary $x = f(z) = \cosh z$ in the xz -plane about the z axis which is called **catenoid**. The Gaussian curvature of catenoid is :

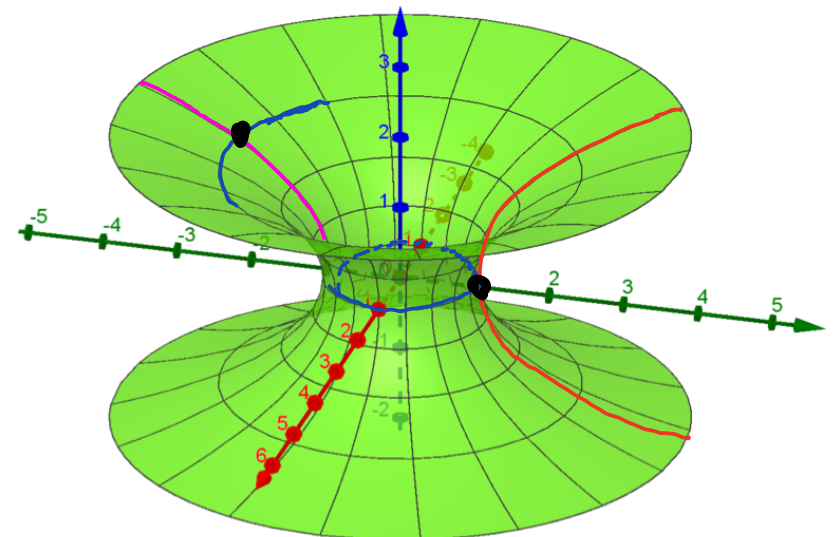
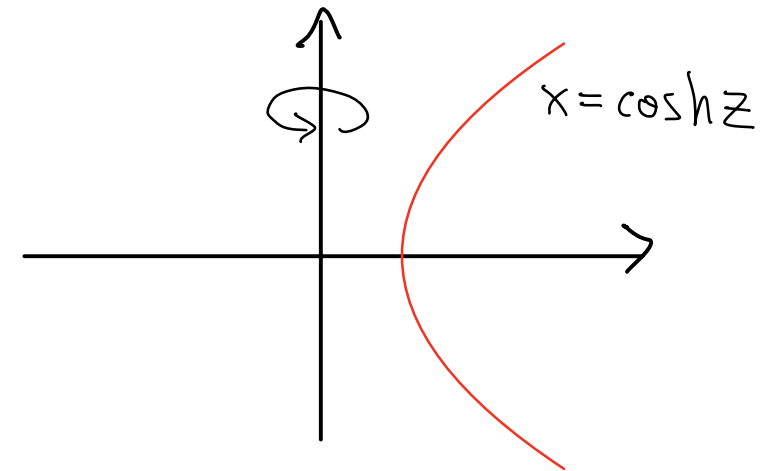
$$f(z) = \cosh z$$

$$f'(z) = \sinh z$$

$$f''(z) = \cosh z$$

$$K(z) = - \frac{\cosh z}{\cosh z (1 + \sinh^2 z)^2}$$
$$= - \frac{1}{\cosh^4 z} < 0$$

$$K(z) = - \frac{f''}{f(1 + f'^2)^2}.$$



Example 3.3.9 (Torus). Show that the Gaussian curvature of the torus obtained by rotating the arc length parametrized curve

$$(x, z) = (\varphi(s), \psi(s)) = \left(R + r \sin \frac{s}{r}, r \cos \frac{s}{r}\right), \quad s \in (0, 2\pi r)$$

about the z -axis is

$$K = \frac{\sin \frac{s}{r}}{r(R + r \sin \frac{s}{r})}.$$

$$K(s) = -\frac{\varphi''}{\varphi}.$$

$$\varphi(s) = R + r \sin \frac{s}{r}$$

$$\varphi'(s) = \cos \frac{s}{r}$$

$$\varphi''(s) = -\frac{1}{r} \sin \frac{s}{r}$$

$$K = -\frac{\varphi''}{\varphi}$$

$$= -\frac{-\frac{1}{r} \sin \frac{s}{r}}{R + r \sin \frac{s}{r}}$$

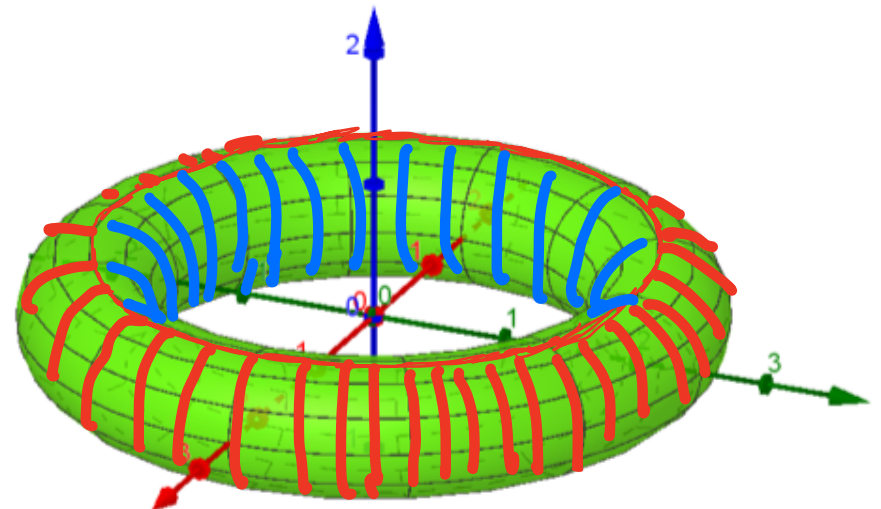
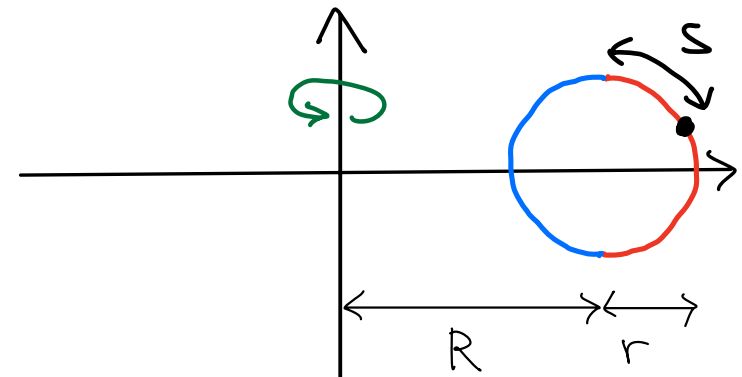
$$= \frac{\sin \frac{s}{r}}{r(R + r \sin \frac{s}{r})}$$

$$K \geq 0$$

"outside"

$$K \leq 0$$

"inside"



Example 3.3.10 (Pseudosphere). Consider the surface obtained by rotating the tractrix (Example 2.2.10)

$$(x, z) = (\varphi(t), \psi(t)) = (\operatorname{sech} t, t - \tanh t), \quad t > 0$$

about the z -axis. This surface is called the **pseudosphere**. Show that the pseudosphere has constant Gaussian curvature equal to -1 .

Method 1:
$$K(u) = \frac{\psi'(\varphi'\psi'' - \varphi''\psi')}{\varphi(\varphi'^2 + \psi'^2)^2}.$$

Method 2: Use arclength parametrization (2.2.10)

$$(\varphi(s), \psi(s)) = (e^{-s}, \ln(e^s + \sqrt{e^{2s} - 1}) - \sqrt{1 - e^{-2s}})$$

$$K = -\frac{\varphi''(s)}{\varphi(s)} = -\frac{e^{-s}}{e^{-s}} = -1$$

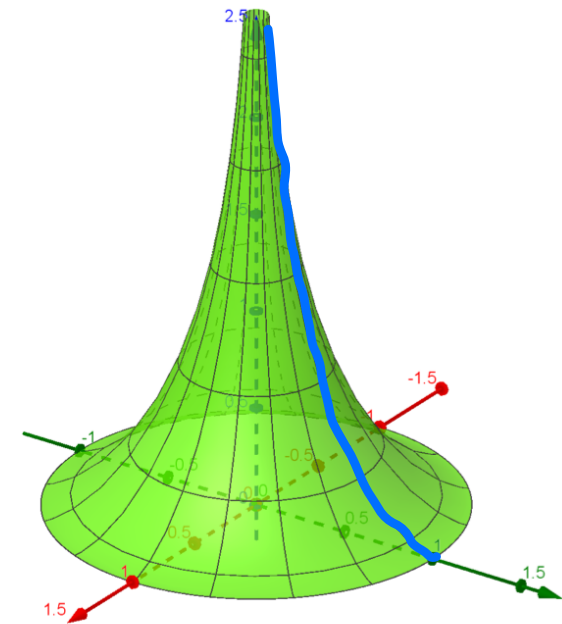
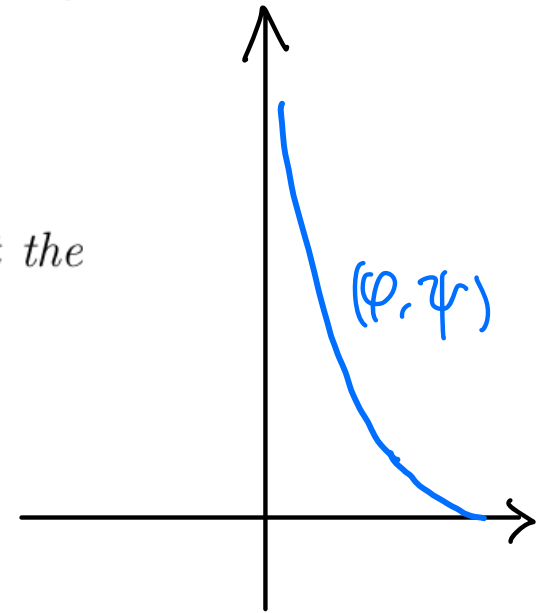


Figure 16: Pseudosphere

Theorem 3.3.11. Let $\mathbf{x}(u, v)$ be a regular parametrized surface. Suppose $F = 0$, i.e., the first fundamental form of $\mathbf{x}(u, v)$ is

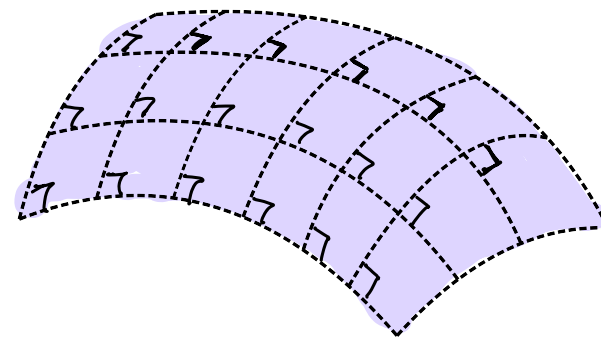
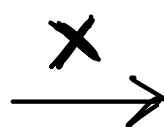
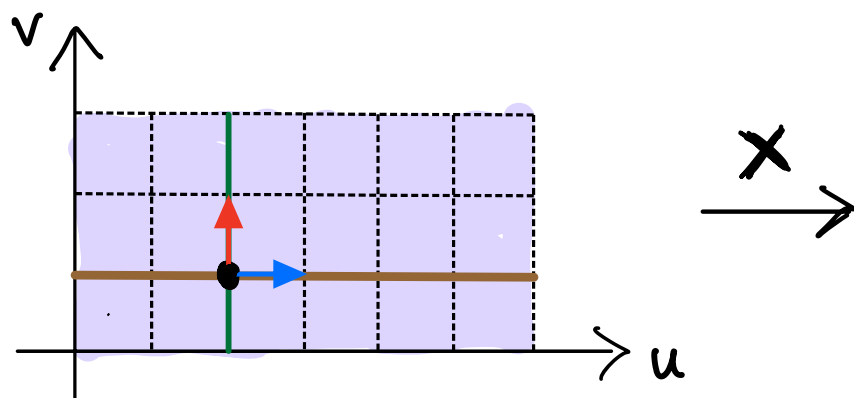
$$I = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}.$$

Then the Gaussian curvature of $\mathbf{x}(u, v)$ is

$$K = -\frac{1}{2\sqrt{EG}} \left[\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right].$$

Pf See Theorem 3.5.4

$$F=0 \Rightarrow \mathbf{x}_u \perp \mathbf{x}_v$$



Example 3.3.12 (Helicoid). Show that the Gaussian curvature of the **helicoid** parametrized by

$$\mathbf{x}(u, \theta) = (u \cos \theta, u \sin \theta, \theta), \quad u, \theta \in \mathbb{R},$$

is

$$K = -\frac{1}{(1+u^2)^2}.$$

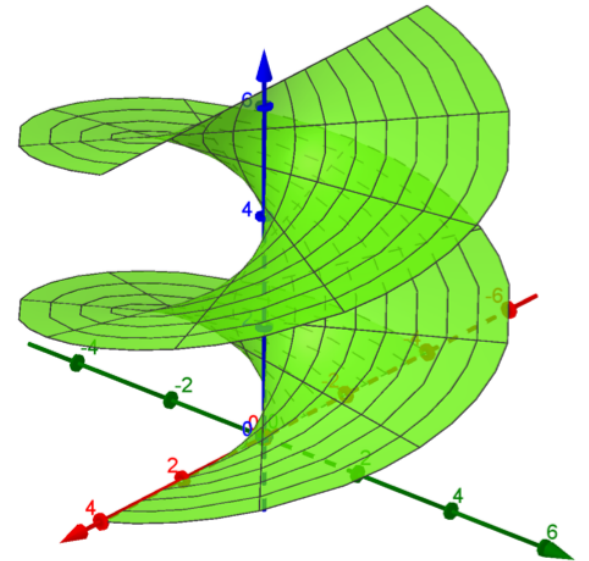
$$X_u = (\cos \theta, \sin \theta, 0) \quad X_\theta = (-u \sin \theta, u \cos \theta, 1)$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1+u^2 \end{bmatrix} \quad E=1 \quad F=0 \quad G=1+u^2$$

$$K = -\frac{1}{2\sqrt{EG}} \left[\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right]$$

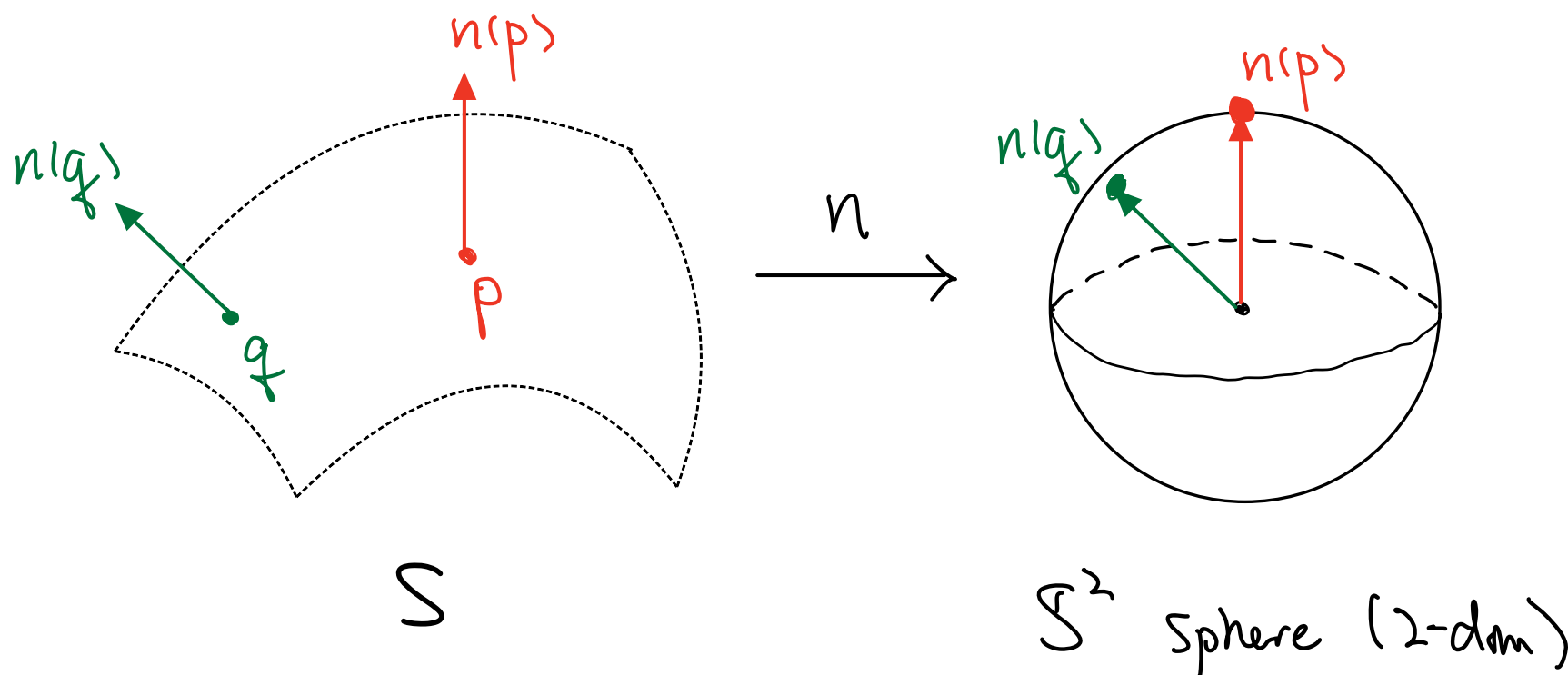
$$= -\frac{1}{2\sqrt{1+u^2}} \left[0 + \left(\frac{2u}{\sqrt{1+u^2}} \right)_u \right]$$

$$= -\frac{1}{2\sqrt{1+u^2}} \cdot \frac{2\sqrt{1+u^2} - 2u \frac{u}{\sqrt{1+u^2}}}{1+u^2} = -\frac{1}{(1+u^2)^2}$$



3.4 Gauss map and its differential

Definition 3.4.1 (Gauss map). Let S be a regular surface in $\mathbb{R}^3$ with regular parametrization $\mathbf{x}(u, v)$. For each $p = \mathbf{x}(u, v)$, we associate the unit normal vector $\mathbf{n}(p)$ to p . This defines a map $\mathbf{n} : S \rightarrow S^2$ from the surface S to the unit sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ and is called the **Gauss map** of S .



Proposition 3.4.2. *Let S be a regular surface with regular parametrization $\mathbf{x}(u, v)$ and $\mathbf{n} : S \rightarrow \mathbb{S}^2$ be the Gauss map which sends a point $p \in S$ to the unit normal vector $\mathbf{n} = \mathbf{n}(p)$ which is a point on the unit sphere $\mathbb{S}^2$. Let $p \in S$ be any point on the surface S . Then the following statements hold.*

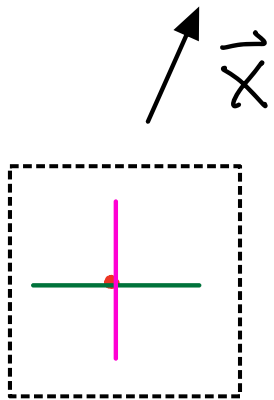
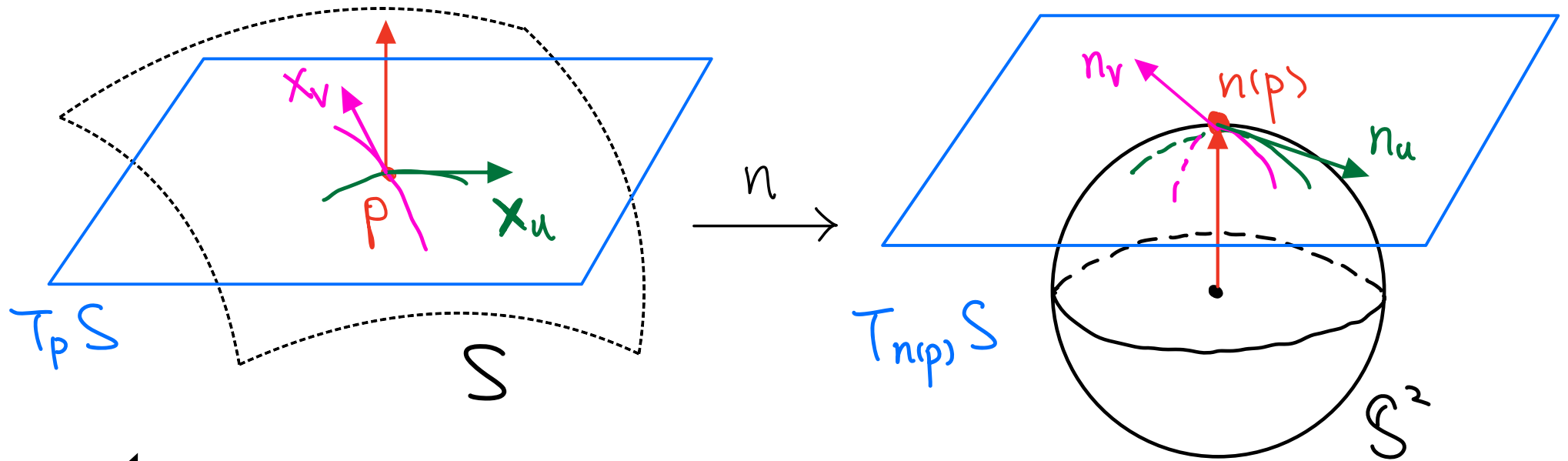
1. *The unit normal vector $\mathbf{n} = \mathbf{n}(p)$ to the surface S is a unit normal vector to the unit sphere $\mathbb{S}^2$ at $\mathbf{n}$.*
2. *The tangent space to the unit sphere $\mathbb{S}^2$ at $\mathbf{n} = \mathbf{n}(p)$ is equal to the tangent space to the surface S at p . In other words,*

$$T_{\mathbf{n}}\mathbb{S}^2 = T_p S.$$

3. *The vectors $\mathbf{n}_u(p)$ and $\mathbf{n}_v(p)$ are tangent to S at p . In other words,*

$$\mathbf{n}_u, \mathbf{n}_v \in T_p S$$

which means both $\mathbf{n}_u, \mathbf{n}_v$ can be written as linear combinations of $\mathbf{x}_u$ and $\mathbf{x}_v$.



- $T_p S = T_{n(p)} S^2 = \{ \vec{v} \in \mathbb{R}^3 : \langle \vec{v}, n(p) \rangle = 0 \}$
- $\|n\| \equiv 1 \Rightarrow \langle n, n \rangle \equiv 1 \Rightarrow \langle n, n \rangle_u = 0$
 $\langle n_u, n \rangle + \langle n, n_u \rangle = 0 \Rightarrow \langle n_u, n \rangle = 0$
Hence $n_u, n_v \in T_{n(p)} S^2 = T_p S$

Later : find a, b, c, d such that $\begin{cases} n_u = ax_u + bx_v \\ n_v = cx_u + dx_v \end{cases}$

Theorem 3.4.3. Let $\mathbf{x}(u, v)$ be a regular parametrized surface and $\mathbf{n}(u, v)$ be the unit normal vector at $\mathbf{x}(u, v)$. Then

$$\mathbf{n}_u \times \mathbf{n}_v = K \mathbf{x}_u \times \mathbf{x}_v$$

where K is the Gaussian curvature of the surface.

Pf $\mathbf{n}_u, \mathbf{n}_v \perp \mathbf{x}_u \times \mathbf{x}_v$

$$\Rightarrow \mathbf{n}_u \times \mathbf{n}_v = c \mathbf{x}_u \times \mathbf{x}_v$$

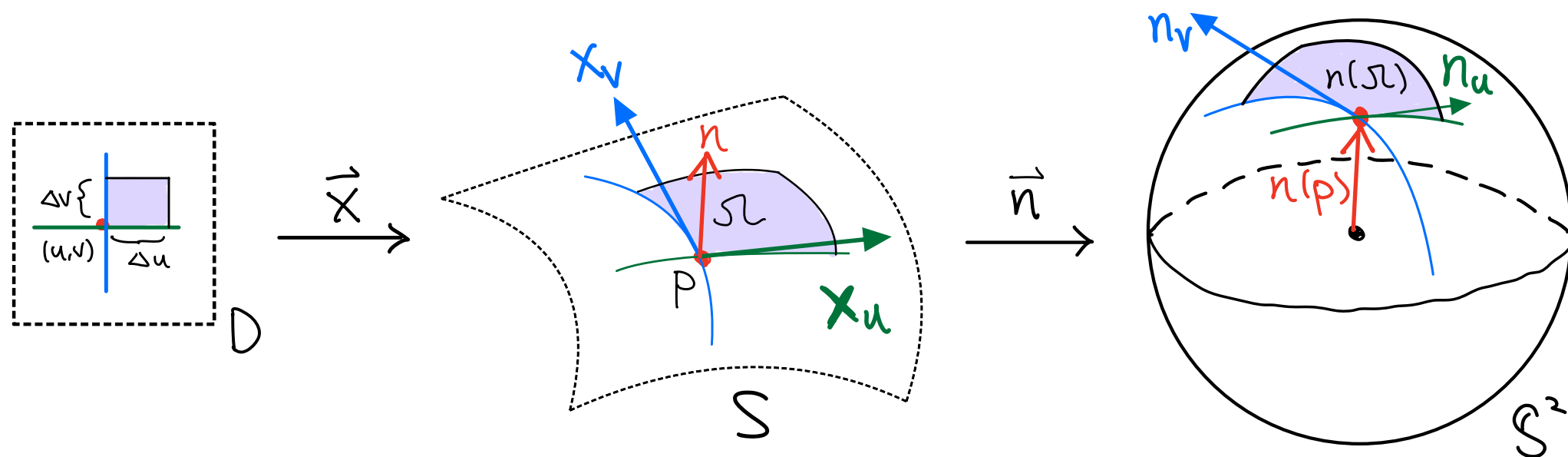
$$\begin{aligned} \det(\mathbb{II}) &= \begin{vmatrix} -\langle \mathbf{x}_u, \mathbf{n}_u \rangle & -\langle \mathbf{x}_u, \mathbf{n}_v \rangle \\ -\langle \mathbf{x}_v, \mathbf{n}_u \rangle & -\langle \mathbf{x}_v, \mathbf{n}_v \rangle \end{vmatrix} \\ &= (-1)^2 \begin{vmatrix} \langle \mathbf{x}_u, \mathbf{n}_u \rangle & \langle \mathbf{x}_u, \mathbf{n}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{n}_u \rangle & \langle \mathbf{x}_v, \mathbf{n}_v \rangle \end{vmatrix} \\ &= \langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{n}_u \times \mathbf{n}_v \rangle \\ &= c \langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_u \times \mathbf{x}_v \rangle \\ &= c \det \mathbb{I} \end{aligned}$$

Hence $c = \frac{\det \mathbb{II}}{\det \mathbb{I}} = K$

Formula

$$\begin{vmatrix} \langle a, c \rangle & \langle a, d \rangle \\ \langle b, c \rangle & \langle b, d \rangle \end{vmatrix} = \langle a \times b, c \times d \rangle$$

Gaussian Curvature as Area ratio



Define $\Omega = \{x(s,t) : u < s < u + \Delta u, v < t < v + \Delta v\} \subset S$

$\Delta A = \text{Area of } \Omega$

$\Delta \sigma = \text{signed area of } n(\Omega) = \begin{cases} \text{Area of } n(\Omega) & \text{if } n_u, n_v, n \text{ satisfy right hand rule} \\ -\text{Area of } n(\Omega) & \text{if } n_u, n_v, n \text{ satisfy left hand rule} \end{cases}$

$$\Delta A \approx \|\Delta u X_u \times \Delta v X_v\| = \|X_u \times X_v\| \Delta u \Delta v$$

orientation

$$|\Delta \sigma| \approx \|\Delta u n_u \times \Delta v n_v\| = \|n_u \times n_v\| \Delta u \Delta v = \|K X_u \times X_v\| \Delta u \Delta v = |K| \Delta A$$

$$\Delta\sigma \approx K \Delta A \text{ or } -K \Delta A ?$$

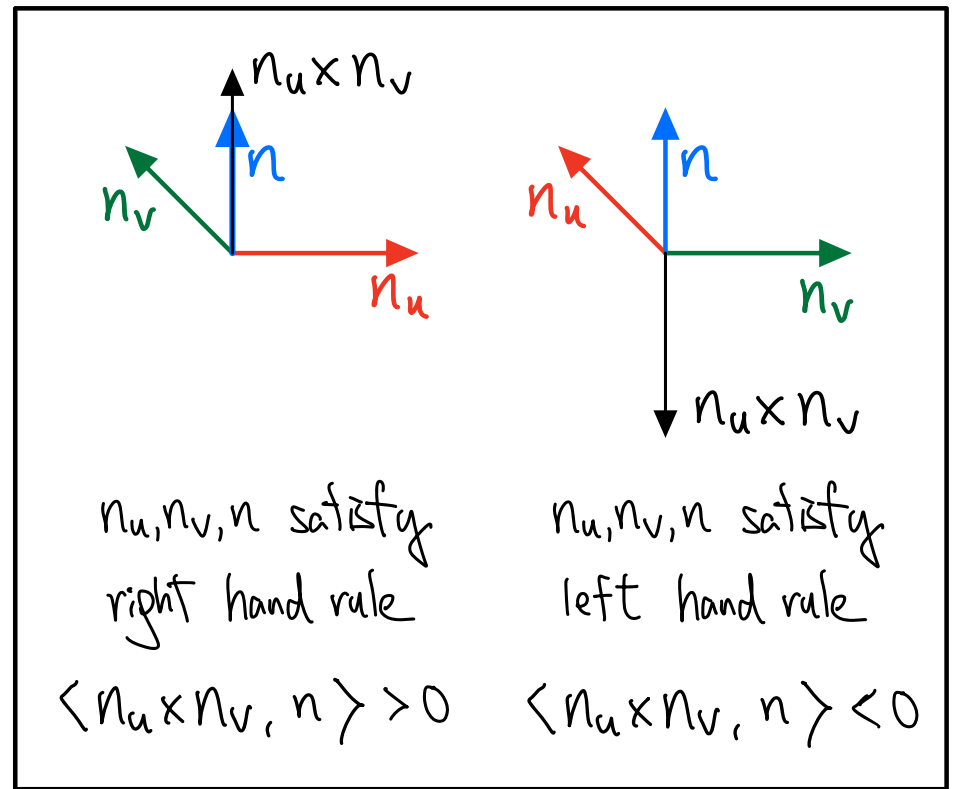
$$\text{sign of } \Delta\sigma = \text{sign of } \langle n_u \times n_v, n \rangle$$

$$\langle n_u \times n_v, n \rangle$$

$$= \left\langle K x_u \times x_v, \frac{x_u \times x_v}{\|x_u \times x_v\|} \right\rangle$$

$$= K \frac{\|x_u \times x_v\|^2}{\|x_u \times x_v\|}$$

$$= K \|x_u \times x_v\| \quad \begin{array}{l} \text{same sign} \\ \text{as } K \end{array}$$



$$\text{Hence } \Delta\sigma \approx K \Delta A, \quad \frac{\Delta\sigma}{\Delta A} \approx K \text{ for small } \Delta u, \Delta v$$

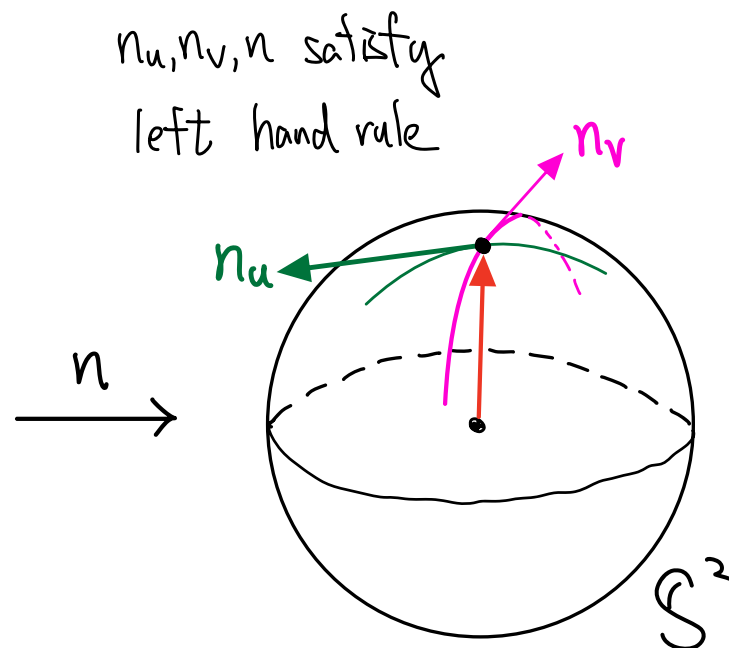
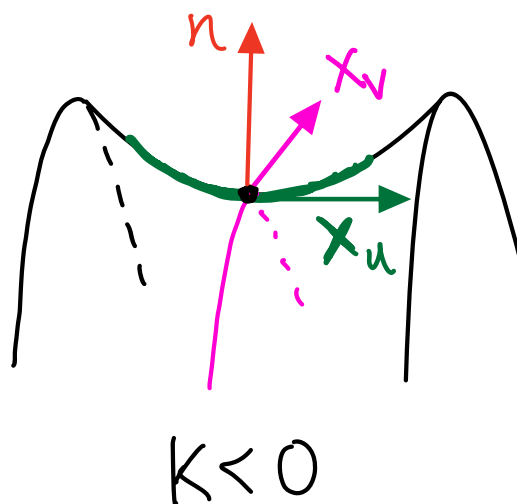
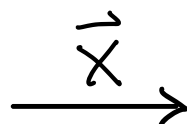
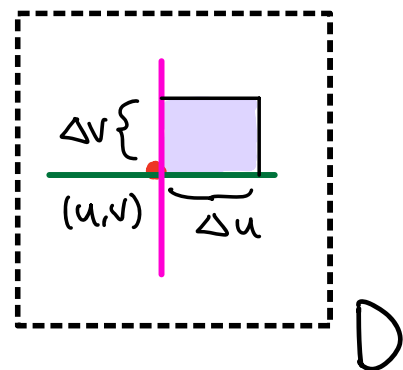
Proposition 3.4.4. Let S be a regular surface with parametrization $\mathbf{x}(u, v)$, $(u, v) \in D$. Let A and σ be the signed surface area function on S and S^2 respectively. Then we have

$$\frac{d\sigma}{dA} = K$$

where K is the Gaussian curvature.

Rmk $\frac{d\sigma}{dA} = \lim_{\Delta A \rightarrow 0} \frac{\Delta\sigma}{\Delta A}$

eg



$\mathbf{n}_u, \mathbf{n}_v, \mathbf{n}$ satisfy
left hand rule