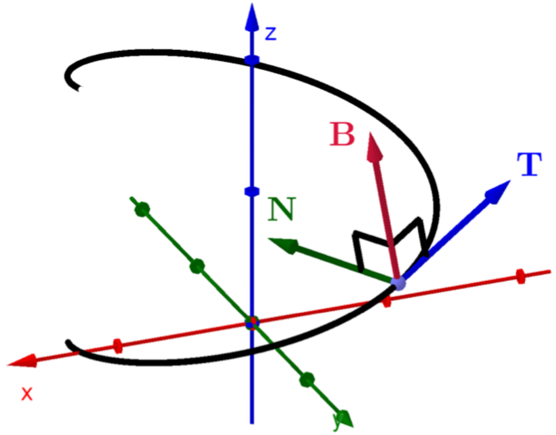
2.4 Frenet frame



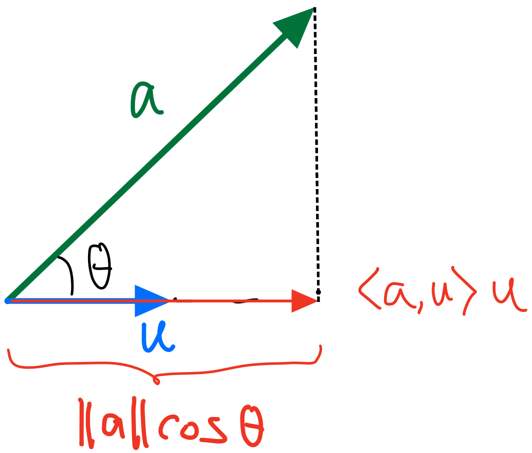
Definition 2.4.1 (Binormal). *Let* $\mathbf{r}(t)$ *be a space curve with curvature* $\kappa(t) > 0$ *for any* t*. We define the unit* **binormal** *to the curve by*

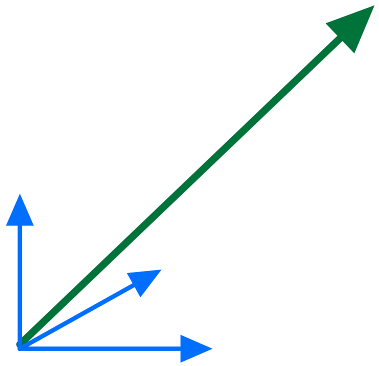
$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

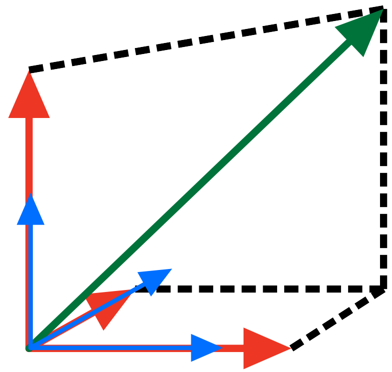
T, N, B are pairwisely orthogonal unit vector in $\mathbb{R}^3$ (called orthonormal basis)

Observation: If $u, v, w \in \mathbb{R}^3$ are orthonormal, then for any $a \in \mathbb{R}^3$,

$a = \langle a,u \rangle u + \langle a,v \rangle v + \langle a,w \rangle w$ (coefficients are unique)







$$\langle a,u \rangle = \|a\| \|u\| \cos\theta = \|a\| \cos\theta$$

Q Let $\vec{r}(s)$ be parametrized by arclength. Express T', N', B' in terms of T, N, B

① $T' = \kappa N = 0T + \kappa N + 0B$ $\qquad N = \frac{T'}{\|T'\|} \qquad \kappa = \frac{\|T'\|}{\|r'\|} = \|T'\|$

② $N' = \langle N', T\rangle T + \langle N', N\rangle N + \langle N', B\rangle B$

$\langle N, N\rangle \equiv 1 \Rightarrow \langle N, N\rangle' = 0 \Rightarrow 2\langle N', N\rangle = 0 \Rightarrow \langle N', N\rangle = 0$

$\langle N, T\rangle \equiv 0 \Rightarrow \langle N, T\rangle' = 0 \Rightarrow \langle N', T\rangle + \langle N, T'\rangle = 0$

$\Rightarrow \langle N', T\rangle = -\langle N, T'\rangle = -\langle N, \kappa N\rangle = -\kappa\langle N, N\rangle = -\kappa$

$\langle N', B\rangle = ?$ New definition:

Definition 2.4.2 (Torsion). *Let* $\mathbf{r}(t)$ *be a space curve with curvature* $\kappa(t) > 0$ *for any* t. *The* **torsion** *of the curve at* $\mathbf{r}(t)$ *is defined by*

$$\tau = \left\langle \frac{d\mathbf{N}}{ds}, \mathbf{B} \right\rangle$$

where s *is a arc length parameter, which means* $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$. *Equivalently, we have*

$$\tau(t) = \left\langle \frac{\mathbf{N}'(t)}{\|\mathbf{r}'(t)\|}, \mathbf{B}(t) \right\rangle$$

Hence, $N' = -\kappa T + 0N + \tau B$

Theorem 2.4.4 (Frenet formula). *Let $\mathbf{r}(s)$ be a regular space curve parametrized by arc length with curvature $\kappa(s) > 0$ for any s. Then*

$$\begin{cases} \mathbf{T}'(s) = \kappa\mathbf{N} & (1) \\ \mathbf{N}'(s) = -\kappa\mathbf{T} + \tau\mathbf{B} & (2) \\ \mathbf{B}'(s) = -\tau\mathbf{N} & (3) \end{cases}$$

discussed

We may write the formula in matrix form

$$\frac{d}{ds}\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}$$

(3) $B' = \langle B', T\rangle T + \langle B', N\rangle N + \langle B', B\rangle B$

$\langle B, B\rangle \equiv 1 \Rightarrow \langle B, B\rangle' \equiv 0 \quad 2\langle B', B\rangle = 0 \Rightarrow \langle B', B\rangle = 0$

$\langle B, T\rangle \equiv 0 \Rightarrow \langle B, T\rangle' \equiv 0 \quad \langle B', T\rangle + \langle B, T'\rangle = 0, \quad \langle B', T\rangle = -\langle B, \kappa N\rangle = 0$

$\langle B, N\rangle \equiv 0 \Rightarrow \langle B', N\rangle = -\langle B, N'\rangle = -\tau$

Proposition 2.4.3. *Let* $\mathbf{r}(t)$ *be a space curve with curvature* $\kappa(t) > 0$ *for any* t. *Then*

$$\tau = \frac{\langle \mathbf{r}' \times \mathbf{r}'', \mathbf{r}''' \rangle}{\|\mathbf{r}' \times \mathbf{r}''\|^2}.$$

$$\begin{cases} \mathbf{T}'(s) &= & \kappa\mathbf{N} & \\ \mathbf{N}'(s) &= -\kappa\mathbf{T} & & +\tau\mathbf{B} \\ \mathbf{B}'(s) &= & -\tau\mathbf{N} & \end{cases}$$

Idea: Express things in term of T, N, B for easier dot/cross product calculation

PF $r' = \|r'\| T$

$r'' = \|r'\|' T + \|r'\| T'$

$= \|r'\|' T + \kappa \|r'\|^2 N$

$T'(t) = \frac{dT}{dt} = \frac{ds}{dt} \cdot \frac{dT}{ds} = \|r'\| (\kappa N)$

$r' \times r'' = \|r'\| \|r'\|' T \times T + \kappa \|r'\|^3 T \times N = \kappa \|r'\|^3 B$

$T \times T = \vec{0}$

$r''' = \|r'\|'' T + \|r'\|' T' + (\kappa \|r'\|^2)' N + \kappa \|r'\|^2 N'$

$\langle r' \times r'', r''' \rangle = \kappa^2 \|r'\|^5 \langle B, \frac{ds}{dt} \frac{dN}{ds} \rangle$

$= \kappa^2 \|r'\|^5 \langle B, \|r'\| \tau B \rangle = \kappa^2 \tau \|r'\|^6$

$\langle B, T \rangle = \langle B, T' \rangle = \langle B, N \rangle = 0$

$\|r' \times r''\|^2 = (\kappa \|r'\|^3)^2 = \kappa^2 \|r'\|^6$

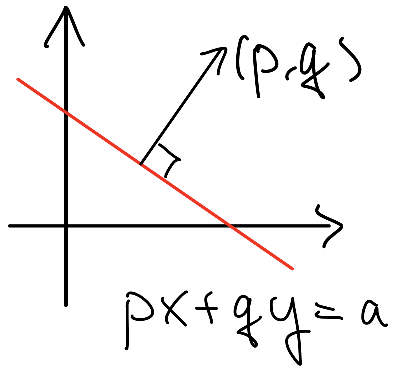
Hence, $\tau = \frac{\langle r' \times r'', r''' \rangle}{\|r' \times r''\|^2}$

Definition 2.4.5 (Plane curve). *We say that a space curve* $\mathbf{r}$ *is a* **plane curve** *if there exists a unit vector* $\mathbf{n}$ *such that*

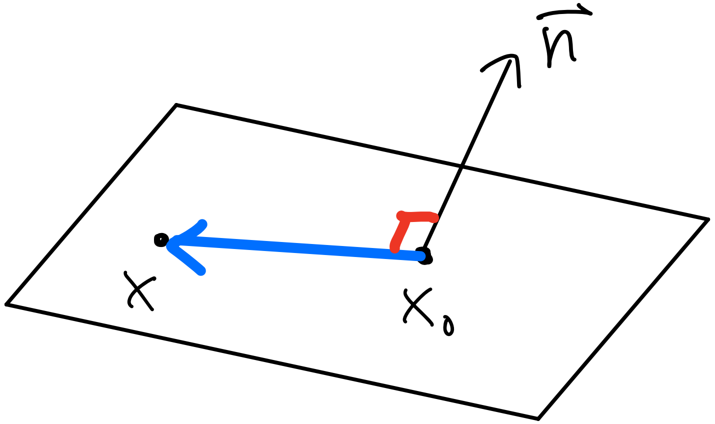
$$\langle \mathbf{r}, \mathbf{n} \rangle = a$$

is a constant.

Analog in $\mathbb{R}^2$



Explanation: Consider a plane in $\mathbb{R}^3$



Suppose $X_0=(x_0,y_0,z_0)$ is a point on it

$\vec{n}=(p,q,r)$ is a normal vector

If $X=(x,y,z)$ is on the plane, then

$$\overrightarrow{X_0X} \perp \vec{n}$$

$$\langle \overrightarrow{X_0X}, \vec{n} \rangle = 0$$

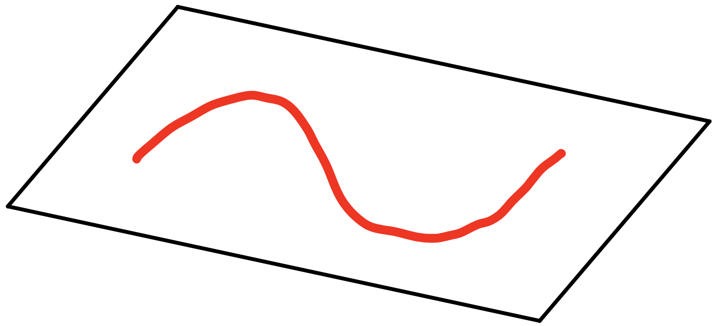
$$\langle (x-x_0, y-y_0, z-z_0), (p,q,r) \rangle = 0$$

$$p(x-x_0)+q(y-y_0)+r(z-z_0)=0$$

$$\Rightarrow \underbrace{px+qy+rz}_{\langle (x,y,z), \vec{n} \rangle} = \underbrace{px_0+qy_0+rz_0}_{a}$$

$\langle r(t), n \rangle = a$ for any t

$\Rightarrow$ $r(t)$ is on the plane for any t



Proposition 2.4.6. *Let $\mathbf{r}(t)$ be a regular parametrized space curve with curvature $\kappa(t) > 0$ for any t. Then $\mathbf{r}$ is a plane curve if and only if its torsion $\tau(t) = 0$ for any t.*

Plane curve $\Leftrightarrow$ $\tau(t) \equiv 0$

Pf $(\Rightarrow)$

$$\langle r, n \rangle = a \quad \forall s$$

$$\frac{d}{dt}: \quad \langle r', n \rangle = 0$$

$$\langle r'', n \rangle = 0$$

$$\langle r''', n \rangle = 0$$

Rmk $n \neq N$

$$\tau = \frac{\langle r' \times r'', r''' \rangle}{\| r' \times r'' \|^2}$$

$r' \times r'' = \alpha \vec{n}$ for some α

$$= \frac{\langle \alpha \vec{n}, r''' \rangle}{\| r' \times r'' \|^2} = 0$$

$(\Leftarrow)$ Assume $\tau(t) \equiv 0$

$$B' = -\tau N = 0$$

$\Rightarrow B \equiv \vec{B_0}$ constant vector

$$\frac{d}{ds} \langle r, \vec{B_0} \rangle = \left\langle \frac{dr}{ds}, \vec{B_0} \right\rangle$$

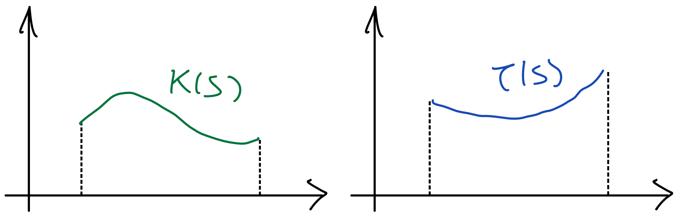
$$= \langle T, \vec{B_0} \rangle$$

$$= \langle T, B \rangle = 0$$

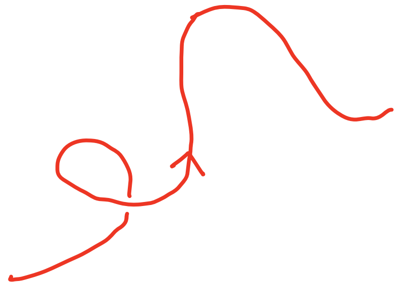
$\langle r, B_0 \rangle = a$ for some constant a

Theorem 2.4.7 (Fundamental theorem of space curves). *Let $\kappa(s), \tau(s) > 0$ be two positive functions. Then there exists unique, up to rigid transformation, space curve $\mathbf{r}(s)$ parametrized by arc length with curvature $\kappa(s)$ and torsion $\tau(s)$.*

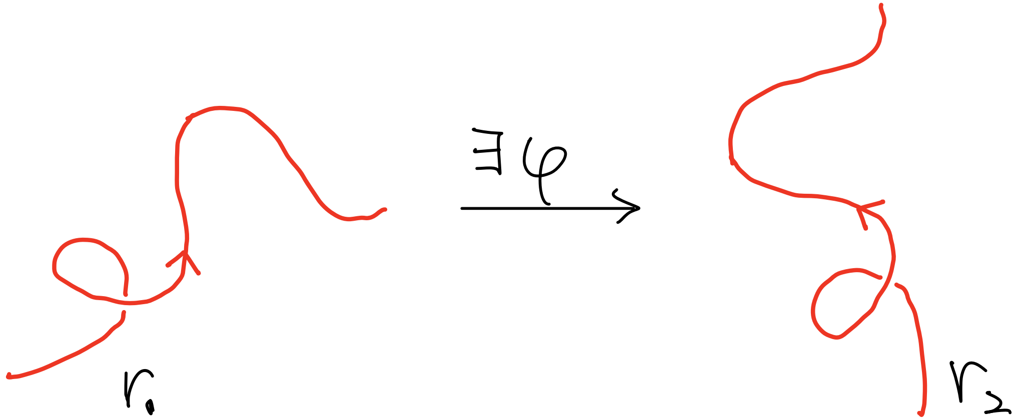
Existence: Any



$\rightsquigarrow$ $\exists\, r(s)$ in $\mathbb{R}^3$ with such κ, τ



Uniqueness: If $r_1(s)$, $r_2(s)$ have same $\kappa(s)$, $\tau(s)$, then there is a rigid transformation (*) $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $r_2(s) = \varphi(r_1(s))$



(*) means $\|\varphi(a) - \varphi(b)\| = \|a - b\|$ for any $a, b \in \mathbb{R}^3$ (Composition of rotation, reflection, translations)

3 Surfaces

① Partial derivatives (derivative with respect to one of the variables)

$f(u,v) = 2u + v^3 + u^2 v$ (two-variable functions)

$f_u = \dfrac{\partial f}{\partial u} = 2 + 2uv$ (Regard v as a constant)

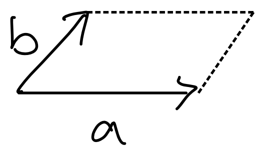
$f_v = \dfrac{\partial f}{\partial v} = 3v^2 + u^2$

$f_{vv} = (f_v)_v = 6v$

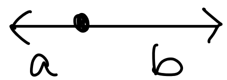
$f_{uu} = 2v = 2v$

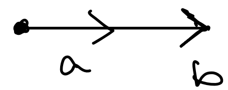
$f_{uv} = (f_u)_v = 2u$

> **Fact** Under mild condition,
>
> $$f_{vu} = f_{uv}$$

② $a, b \in \mathbb{R}^3$ $\quad \vec{a} \times \vec{b} \neq 0$ if 

has non-zero area

$\vec{a} \times \vec{b} = 0$ if $\vec{a} = 0$ or $\vec{b} = 0$ or 

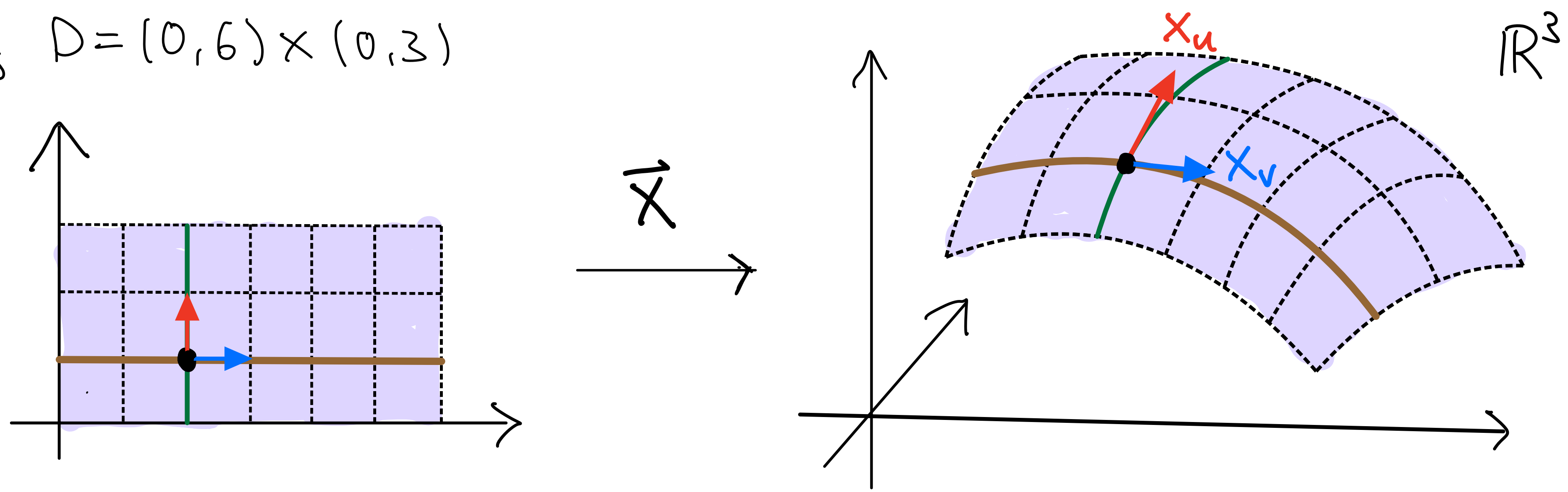
or 

3 Surfaces

3.1 Regular parametrized surfaces

> **Definition 3.1.1** (Regular parametrized surface). *A* **regular parametrized surface** *is a differentiable function* $\mathbf{x} : D \to \mathbb{R}^3$*, where* $D \subset \mathbb{R}^2$ *is an open connected subset, such that* $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$*, for any* $(u, v) \in D \subset \mathbb{R}^2$*. The image* $S = \mathbf{x}(D) \subset \mathbb{R}^3$ *is called a* **regular surface**.

eg $D = (0,6) \times (0,3)$



① For $\mathbf{x}(u, v)$, we denote $\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}$ and $\mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}$ to be the partial derivatives

② $x_u \times x_v \neq 0 \Rightarrow x_u, x_v$ are non-zero and not in the same/opposite directions

Definition 3.1.2 (Tangent space)**.** *Let S be a regular surface with parametrization $\mathbf{x}(u,v)$. The* **tangent space** *of S at $p = \mathbf{x}(u,v)$ is*

$$T_pS = \{\alpha \mathbf{x}_u + \beta \mathbf{x}_v : \alpha, \beta \in \mathbb{R}\} \subset \mathbb{R}^3.$$

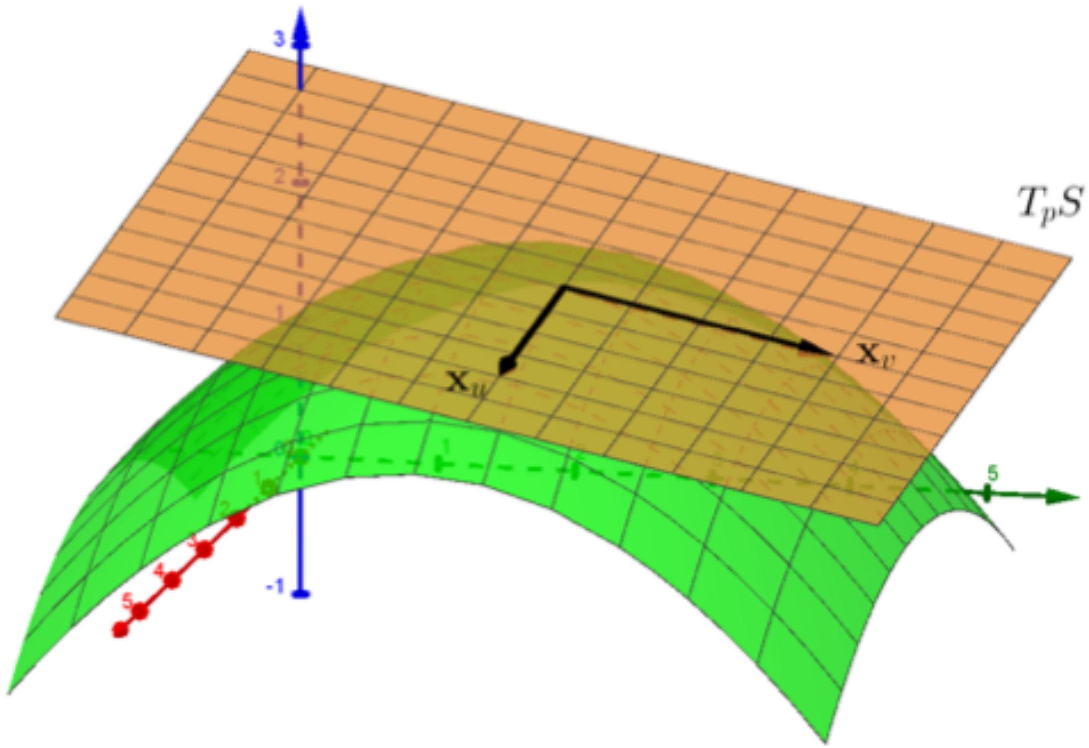
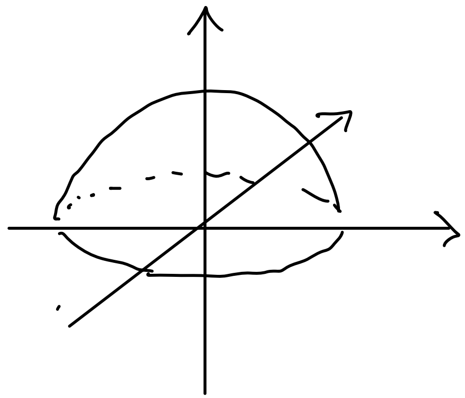


Figure 10: Tangent space

<u>Sphere</u> $\{(x,y,z) : x^2+y^2+z^2 = r^2\}$ hollow

Parametrization? $\vec{x}(x,y) = (x, y, \underbrace{\sqrt{r^2-x^2-y^2}}_{\geq 0 \Rightarrow \text{ only upper hemisphere}}),\ x^2+y^2 \leq r^2$

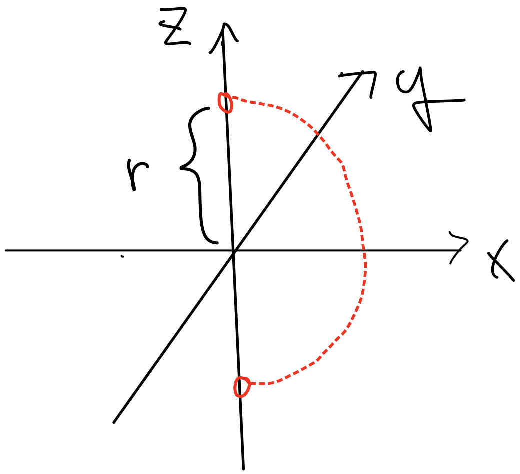


Better parametrization: $\vec{X}(\phi,\theta) = (r\sin\phi\cos\theta,\ r\sin\phi\sin\theta,\ r\cos\phi)$

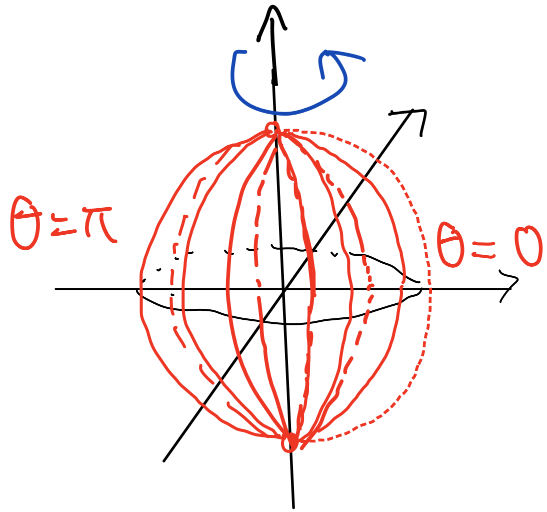
$(\phi,\theta) \in (0,\pi) \times (0,2\pi)$

Take $\theta = 0$, $\phi \in (0,\pi)$

$\vec{X}(\phi,0) = (r\sin\phi,\ 0,\ r\cos\phi)$



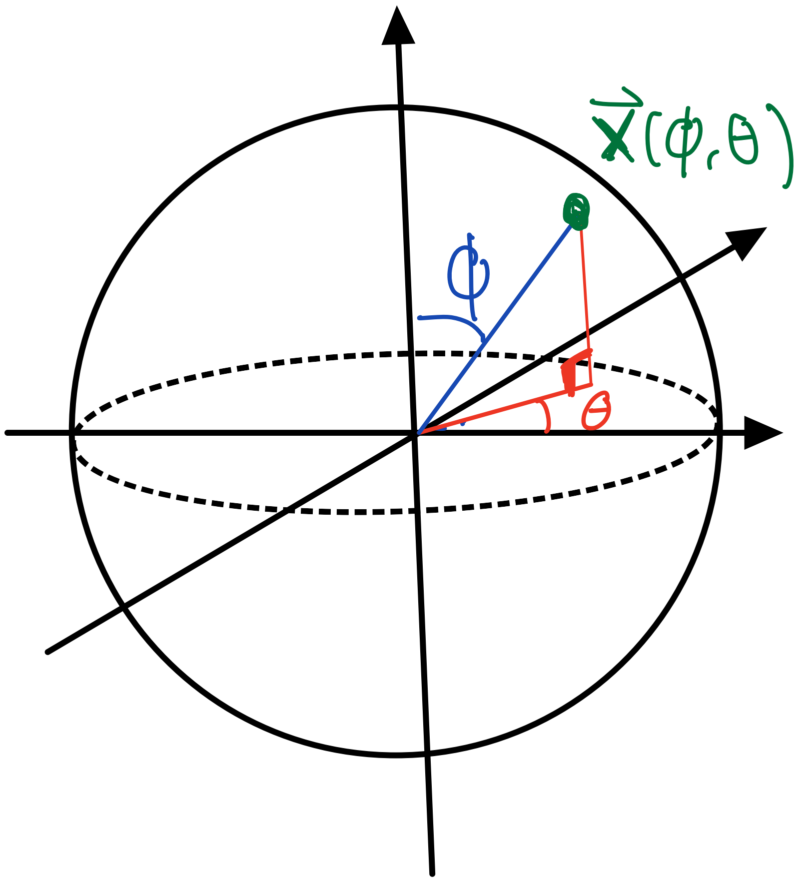
$\xrightarrow[\theta \in (0,2\pi)]{\text{Rotate around } z\text{-axis}}$



1. *Sphere: Let $r > 0$ be a positive real number. The function*

$$\mathbf{x}(\phi, \theta) = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi), \ \textit{for } (\phi, \theta) \in (0, \pi) \times (0, 2\pi)$$

defines a **sphere** *of radius r centered at the origin.*



(r, θ, ϕ) is called Spherical coordinates

2. *Torus: Let $R > r > 0$ be positive real numbers. The function*

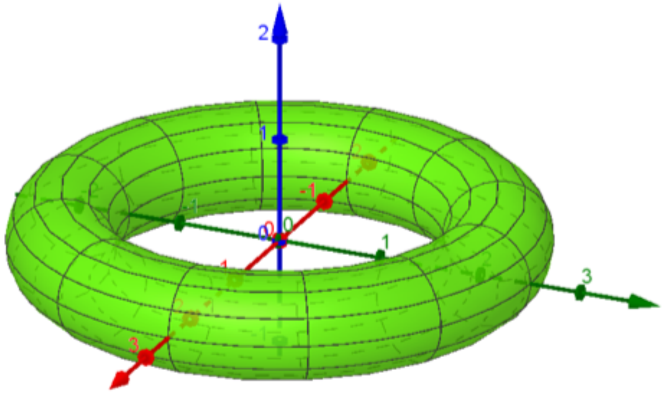
$$\mathbf{x}(\phi,\theta) = ((R+r\sin\phi)\cos\theta, (R+r\sin\phi)\sin\theta, r\cos\phi), \quad \textit{for } \phi,\theta \in (0,2\pi)$$

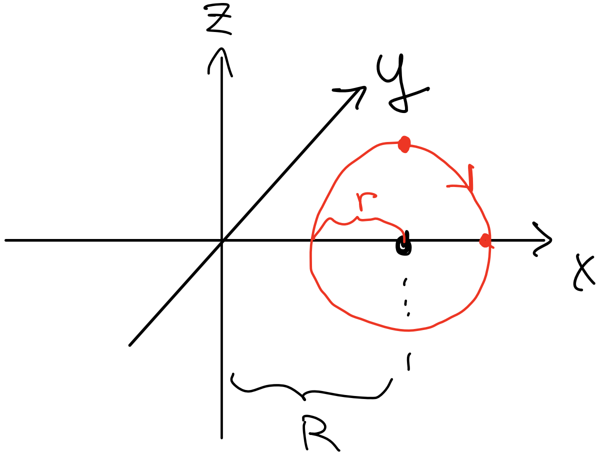
defines a regular surface which is called **torus**.

$R > r > 0$

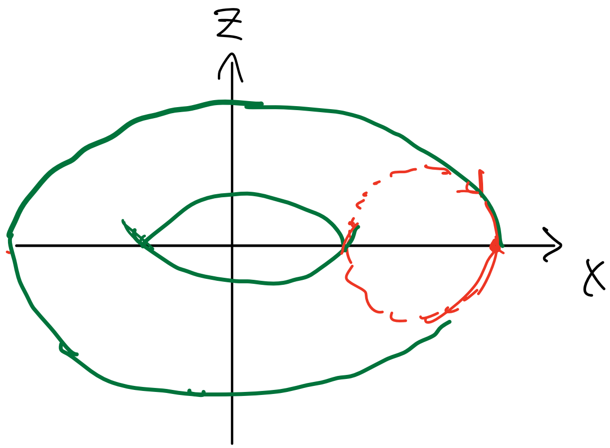
if $\theta = 0$ $\quad X(\phi, 0) = (R + r\sin\phi, 0, r\cos\phi)$

$$= (R,0,0) + r(\sin\phi, 0, \cos\phi)$$





rotate around z-axis



$\theta = 0$

$\phi \in (0, 2\pi)$

$\theta \in (0, 2\pi)$

$\phi \in (0, 2\pi)$

3. *Helicoid: Let $a > 0$ be positive real numbers. The function*

$$\mathbf{x}(u, \theta) = (u\cos\theta, u\sin\theta, a\theta), \ \textit{for } u, \theta \in \mathbb{R}$$

defines a regular surface which is called **helicoid**.