2.2 Arc length

Theorem

~~**Definition**~~ **2.2.1** (Arc length). *Let* $\mathbf{r} : (a,b) \to \mathbb{R}^n$ *be a regular parametrized curve. Then the* **arc length** *of* $\mathbf{r}$ *~~is defined~~ by*

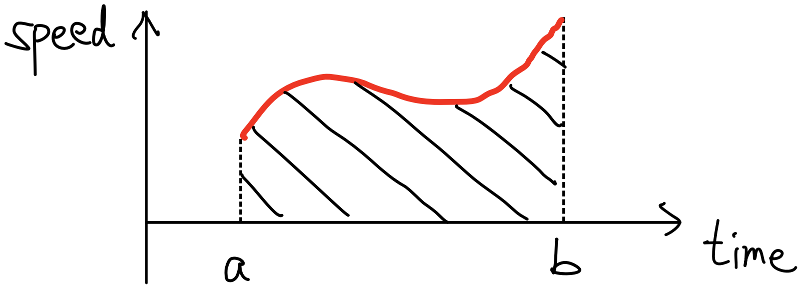
$$l = \int_a^b \|\mathbf{r}'(t)\| dt.$$

Explanation using Physics:

Suppose $r(t)$ = displacement / location at time t

then $r'(t)$ = velocity

$\|r'(t)\|$ = speed

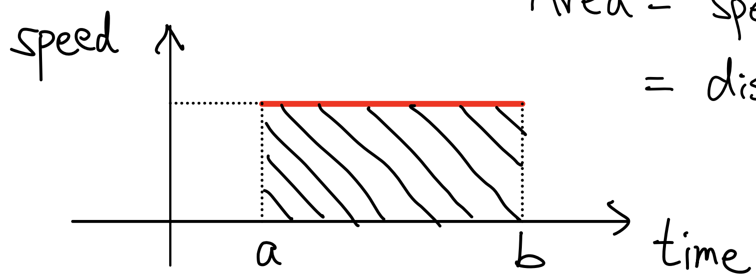


$\int_a^b \|r'(t)\| dt$ = Area = distance travelled

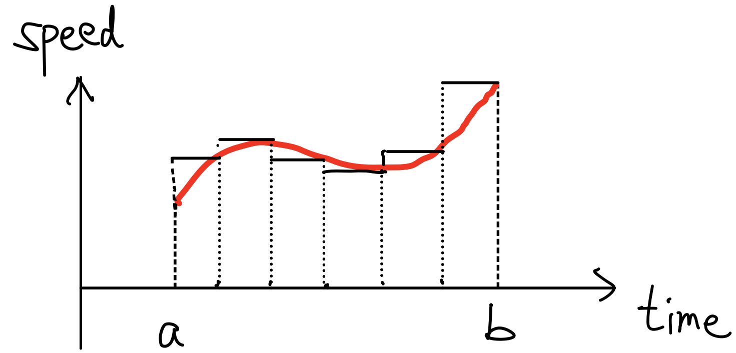
Rmk ① Constant speed

Area = speed × time

= distance



② Riemann sum approximation



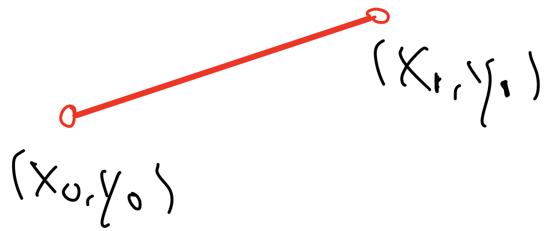
Example 2.2.2 (Arc length of line segments). *Let*

$$\mathbf{r}(t) = ((1-t)x_0 + tx_1, (1-t)y_0 + ty_1), \ 0 < t < 1,$$

$$\boxed{l = \int_a^b \|\mathbf{r}'(t)\| dt.}$$

$$r'(t) = (-x_0 + x_1, -y_0 + y_1)$$

$$\|r'(t)\| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$



$$l = \int_0^1 \|r'(t)\| dt = \int_0^1 \sqrt{(x_1-x_0)^2 + (y_1-y_0)^2}\, dt$$

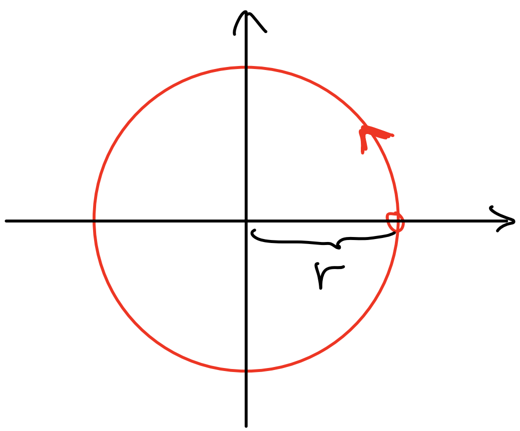
$$= \left[\sqrt{(x_1-x_0)^2 + (y_1-y_0)^2}\, t\right]_0^1 = \sqrt{(x_1-x_0)^2 + (y_1-y_0)^2}$$

Example 2.2.3 (Arc length of circles). *Let* $\mathbf{r}(\theta) = (r\cos\theta, r\sin\theta)$, $0 < \theta < 2\pi$, *be the circle with radius* $r > 0$ *centered at the origin. Now*

$$\vec{r}(\theta) = (r\cos\theta, r\sin\theta)$$

$$\vec{r}'(\theta) = (-r\sin\theta, r\cos\theta)$$

$$\|\vec{r}'(\theta)\| = \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2} = r$$



$$l = \int_0^{2\pi} \|\vec{r}'(\theta)\| d\theta = \int_0^{2\pi} r\, d\theta = \left[r\theta\right]_0^{2\pi} = 2\pi r$$

Proposition 2.2.4 (Arc length of graphs of functions).

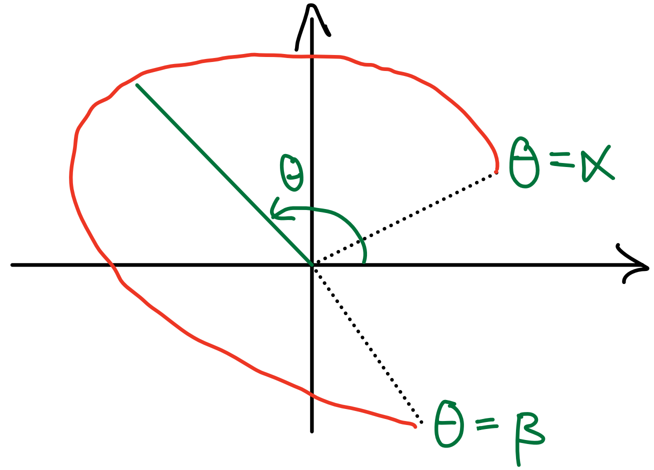
1. *(Rectangular coordinates): The arc length of the curve given by the graph of function $y = f(x)$, $a < x < b$, in rectangular coordinates is*

$$l = \int_a^b \sqrt{1 + f'^2}\,dx.$$

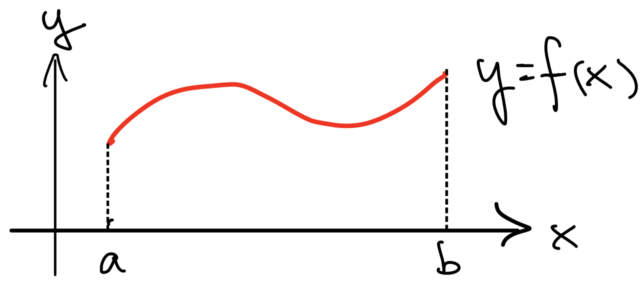
2. *(Polar coordinates): The arc length of the curve given by the graph of function $r = r(\theta)$, $\alpha < \theta < \beta$, in polar coordinates is*

$$l = \int_\alpha^\beta \sqrt{r^2 + r'^2}\,d\theta.$$

$$l = \int_a^b \|\mathbf{r}'(t)\|dt.$$



①



$\vec{r}(x) = (x, f(x))$

$\vec{r}'(x) = (1, f'(x))$

$\|r'(x)\| = \sqrt{1^2 + f'(x)^2}$

$\ell = \int_a^b \|r'(x)\|\,dx$

$= \int_a^b \sqrt{1 + f'(x)^2}\,dx$

②

$x = r(\theta)\cos\theta$

$y = r(\theta)\sin\theta$

$\vec{r}(\theta) = (r\cos\theta, r\sin\theta)$

$\vec{r}'(\theta) = (r'\cos\theta - r\sin\theta, r'\sin\theta + r\cos\theta)$

$\|\vec{r}'(\theta)\|^2 = (r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2$

$= r^2 + r'^2$

$\ell = \int_\alpha^\beta \|r'(\theta)\|\,d\theta = \int_\alpha^\beta \sqrt{r^2 + r'^2}\,d\theta$

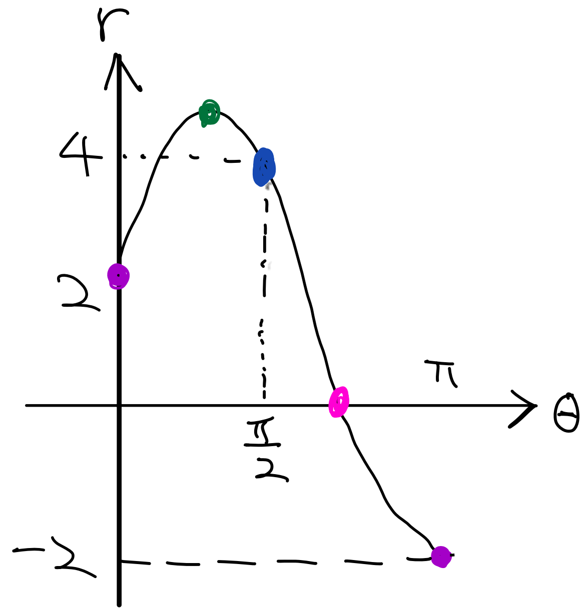
$$l = \int_\alpha^\beta \sqrt{r^2 + r'^2}\, d\theta.$$

eg $\quad r = 2\cos\theta + 4\sin\theta \quad 0 < \theta < \pi$

$r' = -2\sin\theta + 4\cos\theta$

$\sqrt{r^2 + (r')^2} = \sqrt{(2\cos\theta + 4\sin\theta)^2 + (-2\sin\theta + 4\cos\theta)^2} = \sqrt{20}$

$l = \int_0^\pi \sqrt{20}\, d\theta = \sqrt{20}\,\pi$

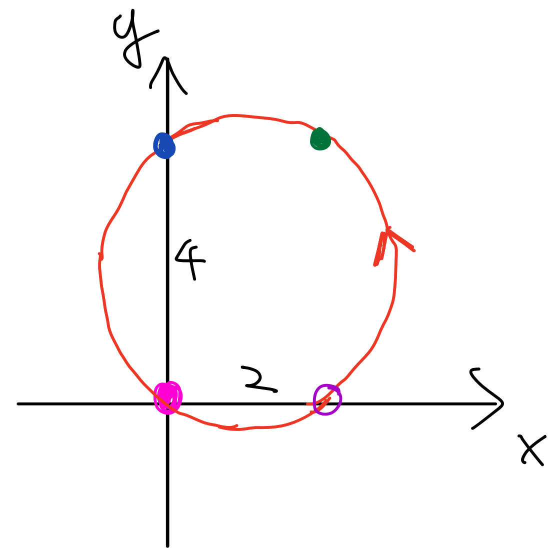


Rmk $\quad r = 2\cos\theta + 4\sin\theta \qquad x = r\cos\theta$

$r^2 = 2r\cos\theta + 4r\sin\theta \qquad y = r\sin\theta$

$x^2 + y^2 = 2x + 4y$

$(x-1)^2 + (y-2)^2 = 5 \qquad$ Circle

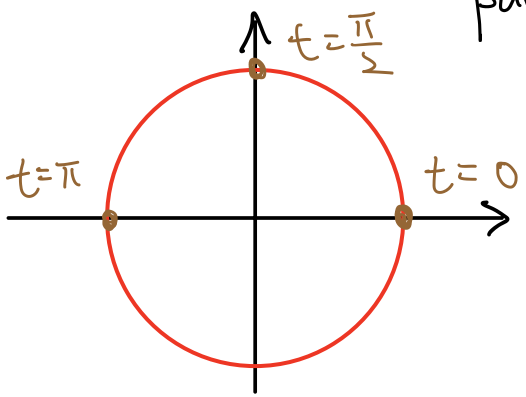


Definition 2.2.5 (Arc length parametrization). *We say that* $\mathbf{r}(s)$ *is an* **arc length parametrized** *curve, or* $\mathbf{r}(s)$ *is* **parametrized by arc length**, *if* $\|\mathbf{r}'(s)\| = 1$ *for any* s.

eg $r(t) = (2\cos t, 2\sin t)$

$r'(t) = (-2\sin t, 2\cos t)$

$\|r'(t)\| = 2$ Not arc length parametrization



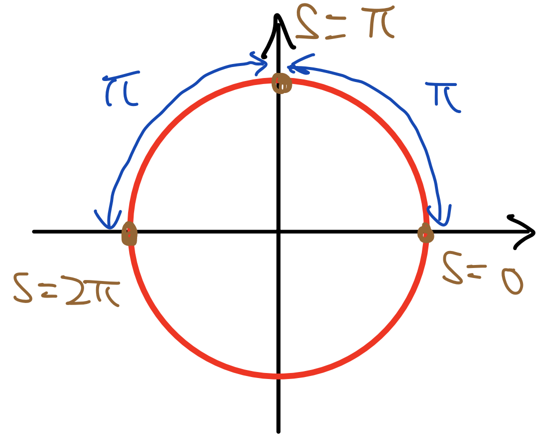
Reparametrize it by arclength? Yes!

$r(s) = (2\cos\frac{s}{2}, 2\sin\frac{s}{2})$

$r'(s) = (-\sin\frac{s}{2}, \cos\frac{s}{2})$

$\|r'(s)\| = 1$

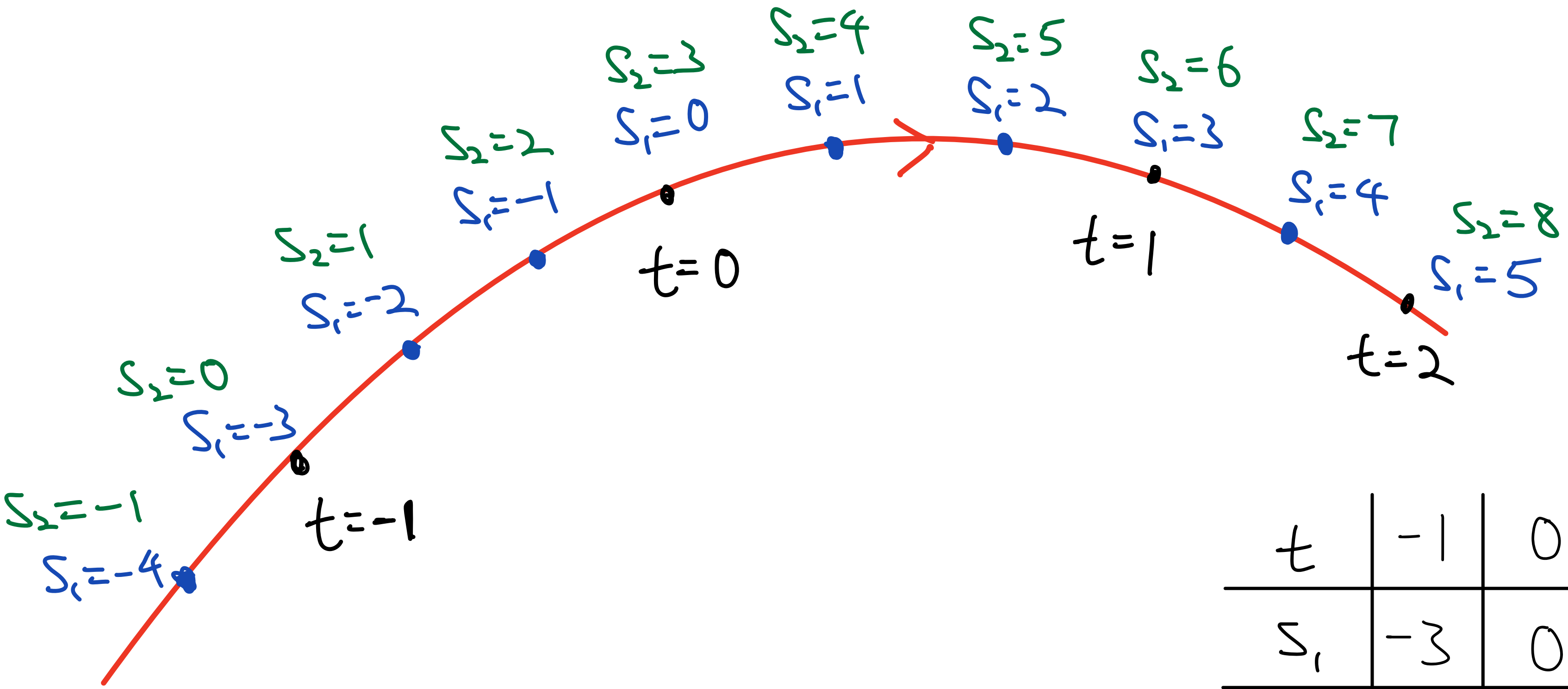
$r(s)$ is an arc length parametrization



Proposition 2.2.6. *Let* $\mathbf{r}(s)$, *be an arc length parametrized curve. Then for* $a < b$, *the arc length of* $\mathbf{r}(s)$ *from* $s = a$ *to* $s = b$ *is* $b - a$.

Pf $\ell = \int_a^b \|r'(t)\| dt = \int_a^b 1 dt = [t]_a^b = b - a$

Theorem 2.2.7 (Existence and uniqueness of arc length parametrization). *Let* $\mathbf{r}(t)$ *be a regular parametrized curve. Then there exists increasing differentiable function* $s = s(t)$ *such that when* $\mathbf{r}(s)$ *is considered as a function of* s*, it is an arc length parametrized curve. Moreover if* $s_1(t)$ *and* $s_2(t)$ *are two such functions, then* $s_2 - s_1$ *is a constant.*



t	-1	0	1	2
S_1	-3	0	3	5
S_2	0	3	6	8
S_2-S_1	3	3	3	3

Theorem 2.2.7 (Existence and uniqueness of arc length parametrization). *Let* $\mathbf{r}(t)$ *be a regular parametrized curve. Then there exists increasing differentiable function* $s = s(t)$ *such that when* $\mathbf{r}(s)$ *is considered as a function of* s*, it is an arc length parametrized curve. Moreover if* $s_1(t)$ *and* $s_2(t)$ *are two such functions, then* $s_2 - s_1$ *is a constant.*

Proof. Let

$$s(t) = \int_\alpha^t \|\mathbf{r}'(u)\| du.$$

By fundamental theorem of calculus, we have $s'(t) = \|\mathbf{r}'(t)\|$. Then when $\mathbf{r}(s)$ is considered as a function of s and by chain rule, we obtain

$$\frac{d\mathbf{r}}{dt} = \frac{ds}{dt}\frac{d\mathbf{r}}{ds} = \|\mathbf{r}'(t)\|\frac{d\mathbf{r}}{ds}.$$

Thus $\dfrac{d\mathbf{r}}{ds}$ is an unit vector which means $\mathbf{r}(s)$ is an arc length parametrization.

Suppose $s_1(t), s_2(t)$ are two increasing differentiable functions such that $\mathbf{r}(s_1)$ and $\mathbf{r}(s_2)$ are arc length parametrizations. Then

$$\frac{ds_2}{dt}\frac{d\mathbf{r}}{ds_2} = \frac{d\mathbf{r}}{dt} = \frac{ds_1}{dt}\frac{d\mathbf{r}}{ds_1}$$

which implies

$$\left|\frac{ds_2}{dt}\right| = \left|\frac{ds_1}{dt}\right|.$$

Since both $s_1(t), s_2(t)$ are increasing functions, we have $\dfrac{ds_2}{dt} = \dfrac{ds_1}{dt}$ and it follows that $s_2 - s_1$ is a constant. $\square$

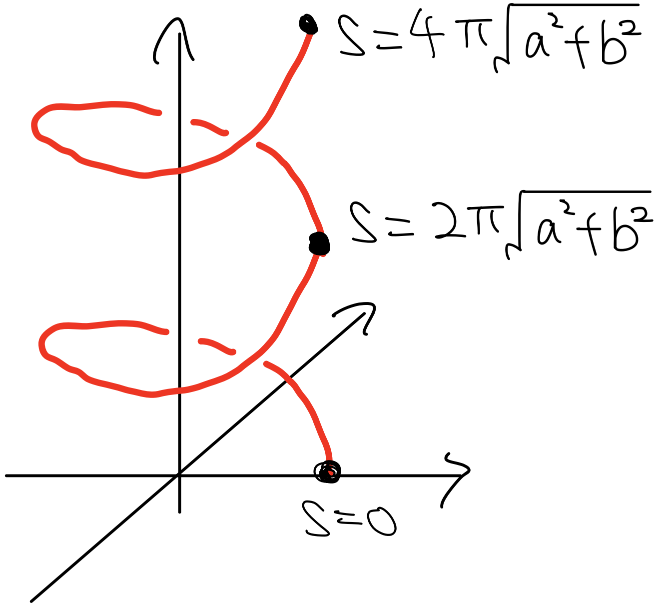
To find the arc length parametrization of $\mathbf{r}(t)$, we do the following three steps.

1. Find the arc length $s(t)$ as a function of t by
$$s(t) = \cancel{\int_a^t \|\mathbf{r}(u)\| du.} \quad \int_a^t \|\mathbf{r}'(u)\| \, du$$
2. Express $t = t(s)$ in terms of s. In other words, make t the subject in $s = s(t)$.
3. Substitute $t(s)$ into t in $\mathbf{r}(t)$ to get the arc length parametrization $\mathbf{r}(s)$.

* We can take a to be any u in the domain of $\vec{r}(u)$

Example 2.2.8 (Arc length parametrization of helix). *Let $a, b > 0$ be constants. Find an arc length parametrization of the helix $\mathbf{r}(\theta) = (a\cos\theta, a\sin\theta, b\theta)$.*

1. $$s(\theta) = \int_0^\theta \|r'(u)\| du$$
$$= \int_0^\theta \sqrt{(-a\sin u)^2 + (a\cos u)^2 + b^2}\, du$$
$$= \int_0^\theta \sqrt{a^2+b^2}\, du = \sqrt{a^2+b^2}\,\theta$$ (s in terms of θ)



2. $$\theta = \frac{s}{\sqrt{a^2+b^2}}$$

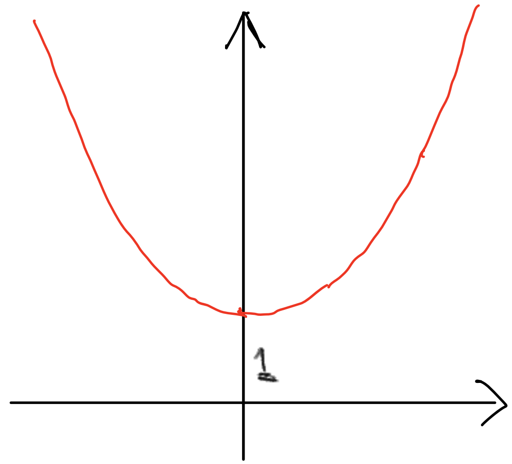
3. $$r(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{bs}{\sqrt{a^2+b^2}} \right)$$

Example 2.2.9 (Arc length parametrization of catenary). *Find an arc length parametrization of the catenary* $\mathbf{r}(t) = (t, \cosh t)$.

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

① $r'(t) = (1, \sinh t)$

$$s(t) = \int_0^t \|r'(u)\| du$$
$$= \int_0^t \sqrt{1^2 + (\sinh u)^2}\, du$$
$$= \int_0^t \cosh u \, du$$
$$= \left[\sinh x\right]_0^t = \sinh t$$



$$s = \frac{e^t - e^{-t}}{2}$$
$$2s = e^t - e^{-t}$$
$$2se^t = e^{2t} - 1$$
$$(e^t)^2 - 2se^t - 1 = 0$$
$$(e^t - s)^2 - s^2 - 1 = 0$$
$$e^t - s = \pm\sqrt{s^2+1}$$

② $t = \sinh^{-1} s = \ln(s + \sqrt{s^2+1})$ $\longleftarrow$

(Any difference if we pick $-\sqrt{s^2+1}$)

③ $r(s) = \left(\ln(s+\sqrt{s^2+1}), \cosh t\right)$
$$= \left(\ln(s+\sqrt{s^2+1}), \sqrt{1+(\sinh t)^2}\right)$$
$$= \left(\ln(s+\sqrt{s^2+1}), \sqrt{1+s^2}\right)$$

$$1 + (\sinh x)^2 = (\cosh x)^2$$

$$\int \cosh x \, dx = \sinh x + C$$

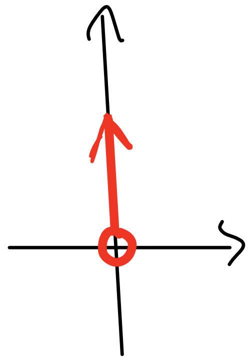
Example 2.2.10 (Tractrix). *The* **tractrix** *is a curve parametrized by*

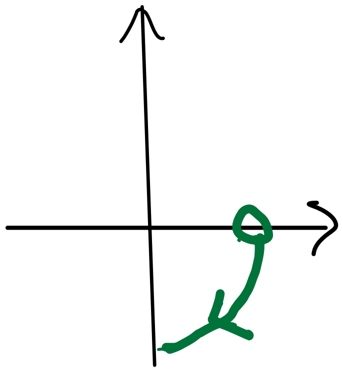
$$\mathbf{r}(t) = (\operatorname{sech} t, t - \tanh t), \quad t > 0.$$

$$= (0, t) + (\operatorname{sech} t, -\tanh t)$$

$$(\cosh t)^2 - (\sinh t)^2 = 1$$

$$(\operatorname{sech} t)^2 + (\tanh t)^2 = 1$$





① $r'(t) = (-\operatorname{sech} t \tanh t, 1 - \operatorname{sech}^2 t)$

$$= (-\operatorname{sech} t \tanh t, \tanh^2 t)$$

$$\|r'(t)\|^2 = \operatorname{sech}^2 t \tanh^2 t + \tanh^4 t$$

$$= \tanh^2 t (\tanh^2 t + \operatorname{sech}^2 t)$$

$$= \tanh^2 t$$

$$S = \int_0^t \|r'(u)\| du$$

$$= \int_0^t \tanh u \, du$$

$$= [\ln \cosh u]_0^t = \ln \cosh t$$

$$\int \frac{\sinh u}{\cosh u} du$$

$$= \int \frac{d \cosh u}{\cosh u}$$

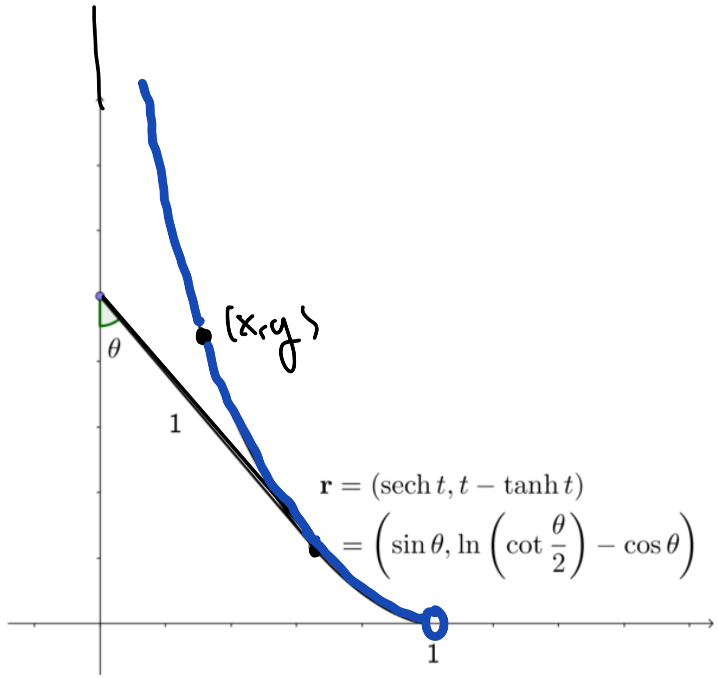


Figure 6: Tractrix

② $\cosh t = e^s$

$\vdots$

$$r(s) = \left(e^{-s}, \ln(e^s + \sqrt{e^{2s} - 1}) - \sqrt{1 - e^{-2s}}\right)$$

Note: The tractrix may also be parametrized by

$$\mathbf{r}(\theta) = \left(\sin\theta, \ln\left(\cot\frac{\theta}{2}\right) - \cos\theta\right), \ 0 < \theta < \frac{\pi}{2}.$$

Suppose L is the tangent to the tractrix at $\mathbf{r}(\theta)$ and P is the point of intersection of L and the y-axis. Then the angle between L and the y-axis is θ and the distance between $\mathbf{r}(\theta)$ and P is always 1.

Three different parametrizations:

$\mathbf{r}(t) = (\operatorname{sech} t, t - \tanh t), \ t > 0.$

$\mathbf{r}(s) = (e^{-s}, \ln(e^s + \sqrt{e^{2s} - 1}) - \sqrt{1 - e^{-2s}}), \ s > 0.$

$\mathbf{r}(\theta) = \left(\sin\theta, \ln\left(\cot\frac{\theta}{2}\right) - \cos\theta\right), \ 0 < \theta < \frac{\pi}{2}.$

To find $y = y(x)$ satisfying this:

$x^2 + (y-b)^2 = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{y-b}{x} = -\frac{\sqrt{1-x^2}}{x}$$

Let $x = \sin\theta$

$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta} = -\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\cdot\cos\theta = -\frac{\cos^2\theta}{\sin\theta}$$

$$\frac{dy}{d\theta} = \frac{\sin^2\theta - 1}{\sin\theta} = \sin\theta - \csc\theta$$

$$y = -\cos\theta - \ln\left|\tan\frac{\theta}{2}\right| + C$$

$y \to 0$ as $\theta \to \frac{\pi}{2} \Rightarrow C = 0$

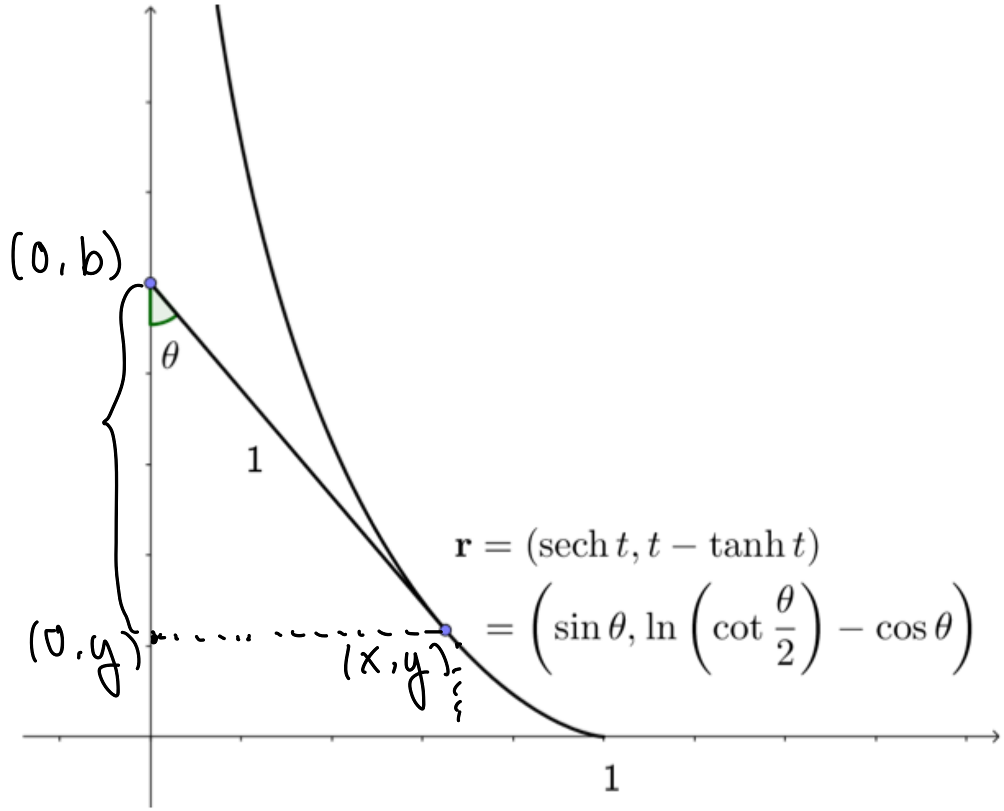


Figure 6: Tractrix

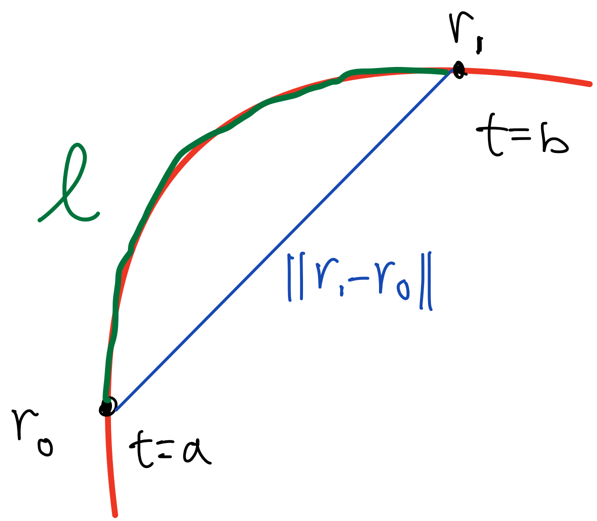
Theorem 2.2.11. *Let $\mathbf{r}(t)$ be a regular parametrized curve with $\mathbf{r}(a) = \mathbf{r}_0$ and $\mathbf{r}(b) = \mathbf{r}_1$. Then the arc length l of the curve from $t = a$ to $t = b$ satisfies*

$$l \geq \|\mathbf{r}_1 - \mathbf{r}_0\|$$

with equality holds if and only if $\mathbf{r}(t)$ is a line segment joining $\mathbf{r}_0$ and $\mathbf{r}_1$.

Pf of inequality

let $\vec{u} = \dfrac{r_1 - r_0}{\|r_1 - r_0\|}$ unit vector in the direction of r_0 to r_1



$$\begin{aligned}
\ell = \int_a^b \|r'(t)\|\,dt &\geq \int_a^b \langle r'(t), u\rangle\,dt \\
&= \int_a^b \langle r(t), u\rangle'\,dt \\
&= \Big[\langle r(t), u\rangle\Big]_a^b \\
&= \langle r(b) - r(a), u\rangle \\
&= \left\langle r_1 - r_0, \frac{r_1 - r_0}{\|r_1 - r_0\|}\right\rangle \\
&= \|r_1 - r_0\|
\end{aligned}$$

Cauchy-Schwarz Inequality

$$|\langle \vec{a}, \vec{b}\rangle| \leq \|\vec{a}\|\,\|\vec{b}\|$$

$$\begin{aligned}
\langle r'(t), u\rangle &\leq \|r'(t)\|\,\|u\| \\
&= \|r'(t)\|
\end{aligned}$$