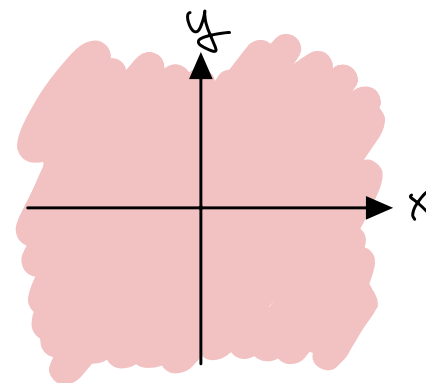
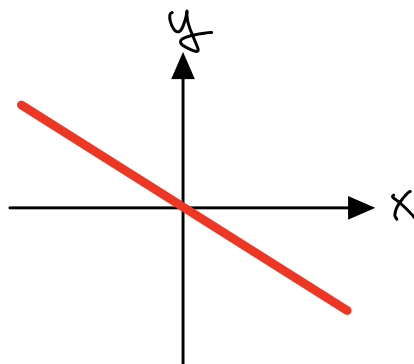
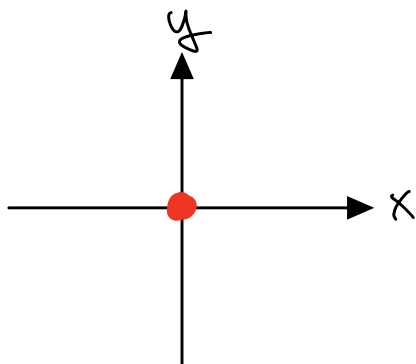


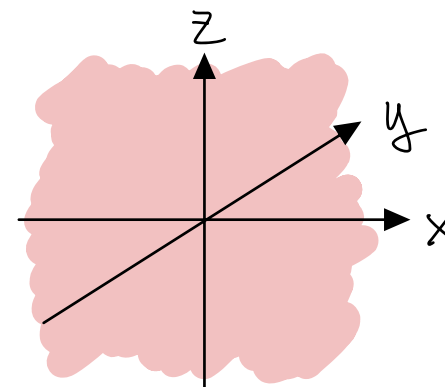
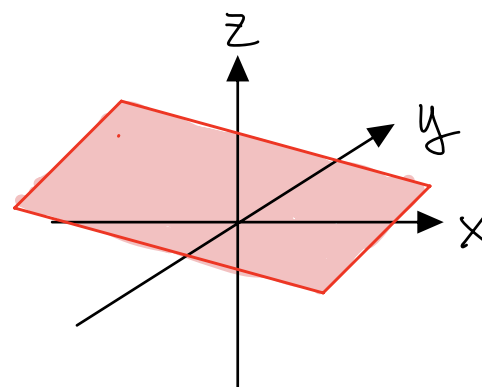
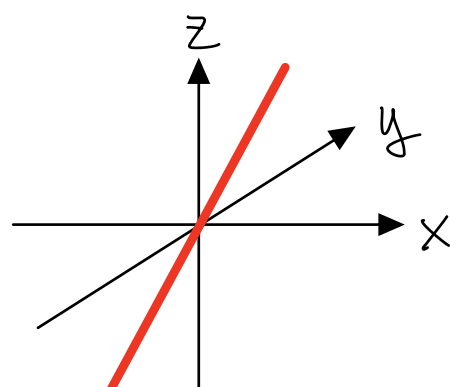
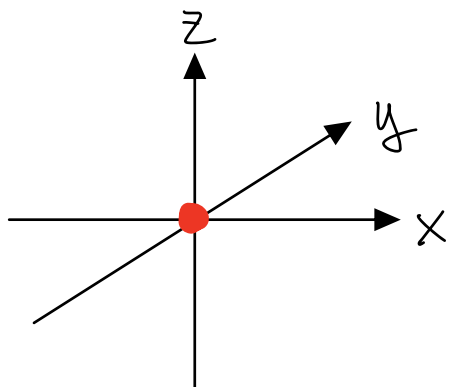
Definition 1.3.18 (Vector subspace). We say that a subset $V \subset \mathbb{R}^m$ is a **vector subspace** of $\mathbb{R}^m$ if V contains the zero vector $\mathbf{0}$ and for any $\mathbf{u}, \mathbf{v} \in V$, $\alpha, \beta \in \mathbb{R}$, we have

$$\alpha \mathbf{u} + \beta \mathbf{v} \in V.$$

eg vector subspaces of $\mathbb{R}^2$



eg vector subspaces of $\mathbb{R}^3$



Definition 1.3.19 (Linearly independent vectors and spanning set). Let $V \subset \mathbb{R}^m$ be a vector subspace and $E = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subset V$ be a set of vectors in V .

1. We say that E is **linearly independent** if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

implies $c_1 = c_2 = \cdots = c_k = 0$.

2. We say that E **spans** V if for any $\mathbf{v} \in V$, there exists scalars $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$ such that

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k.$$

Definition 1.3.23 (Basis). Let $V \subset \mathbb{R}^m$ be a vector subspace and $E = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subset V$ be a set of vectors in V . We say that E constitutes a **basis** for V if

1. E is linearly independent, and
2. E spans V .

Definition 1.3.28 (Dimension). Let V be a vector subspace of $\mathbb{R}^m$. The **dimension** of V is the number of vectors in a basis for V and is denoted by $\dim(V)$.

Theorem 1.3.25. Let $V \subset \mathbb{R}^m$ be a vector subspace and $E = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subset V$ be a set of vectors in V . Then the following conditions are equivalent.

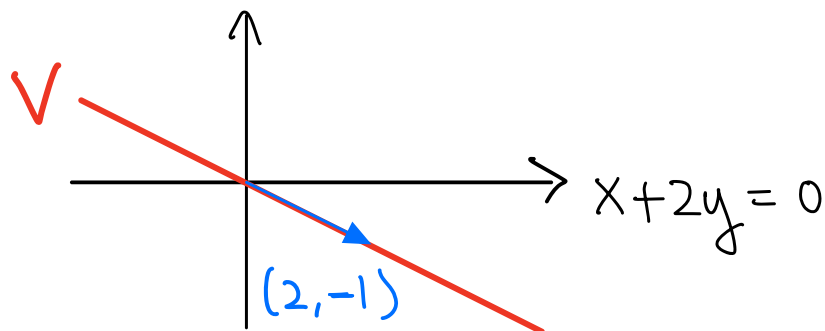
1. E constitutes a basis for V .
2. For any $\mathbf{v} \in V$, there exists unique $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ such that

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n.$$

- If V is a line ($\dim V = 1$) Any basis of V consists of 1 vector

$$E = \{\mathbf{v}_i\}, \text{ where } \mathbf{v}_i \in V, \mathbf{v}_i \neq \mathbf{0}$$

eg



$$E = \{(2, -1)\} \text{ is a basis}$$

For any $\vec{v} \in V$, there exist a unique α such that $\vec{v} = \alpha(2, -1)$

$$\text{eg } (6, -3) = 3(2, -1)$$

$$(-4, 2) = -2(2, -1)$$

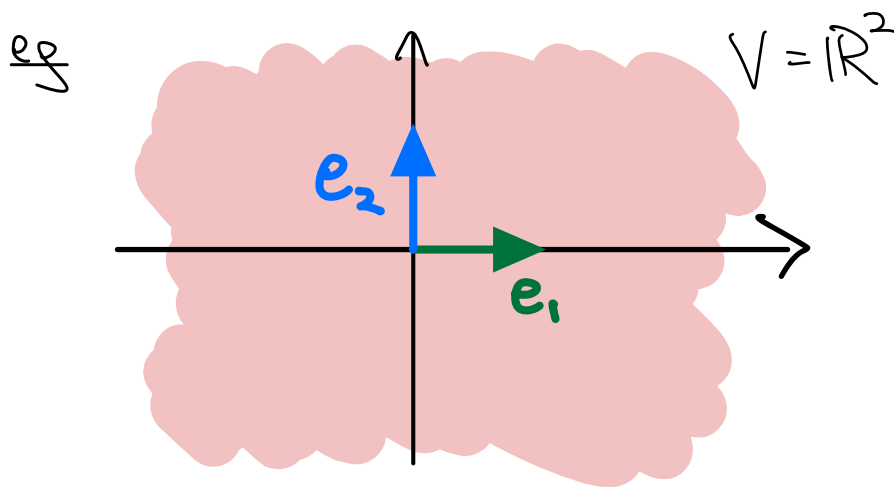
$$(0, 0) = 0(2, -1)$$

Theorem 1.3.25. Let $V \subset \mathbb{R}^m$ be a vector subspace and $E = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subset V$ be a set of vectors in V . Then the following conditions are equivalent.

1. E constitutes a basis for V .
2. For any $\mathbf{v} \in V$, there exists unique $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ such that

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n.$$

- If V is a plane ($\dim V = 2$) Any basis of V consists of 2 vectors $E = \{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1, \mathbf{v}_2 \in V$ are non-zero and not in same/opposite direction



$E = \{e_1, e_2\}$ is a basis
where $e_1 = (1, 0)$ $e_2 = (0, 1)$

For any $\vec{v} \in V$, there exist a unique α, β such that $\vec{v} = \alpha e_1 + \beta e_2$

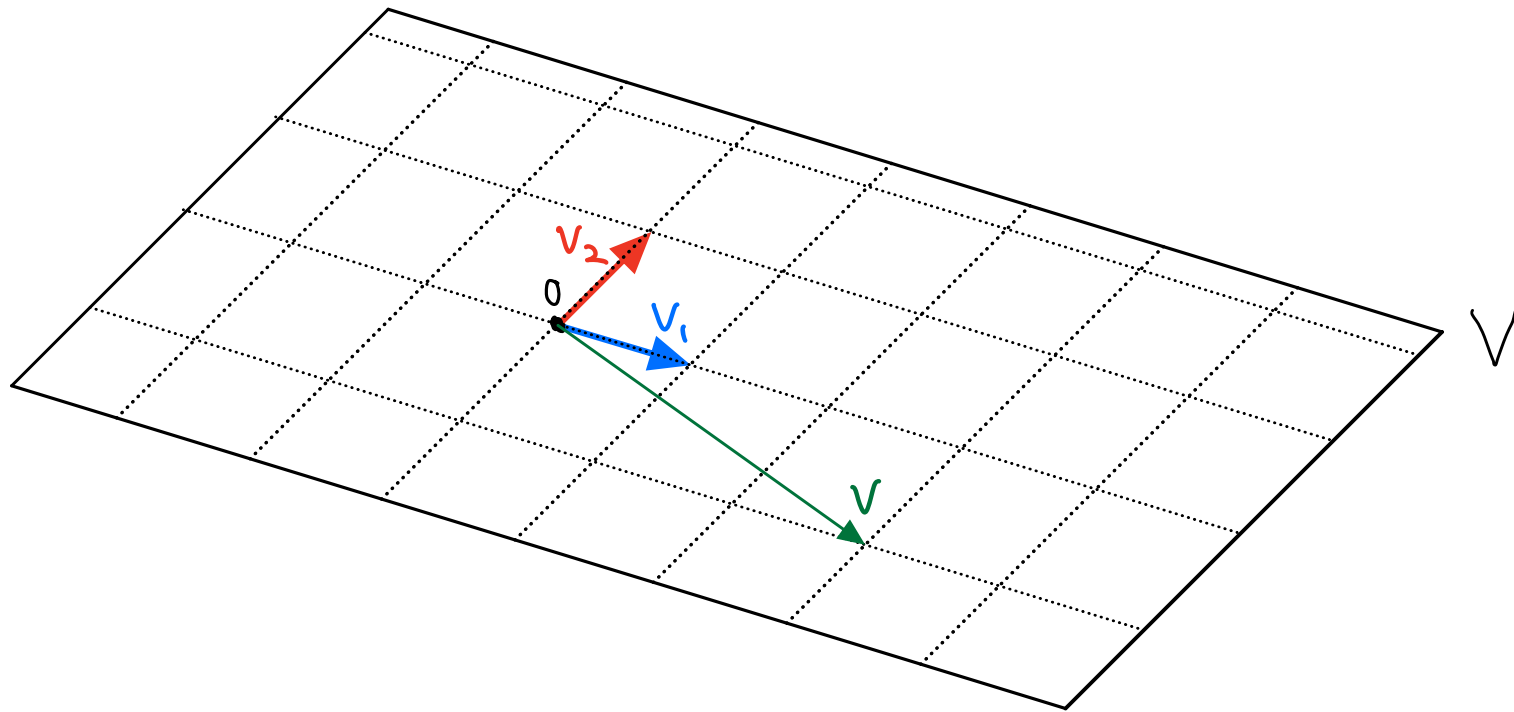
eg $(2, 1) = 2e_1 + 1e_2$

In general, for any $(x, y) \in V$
 $(x, y) = x e_1 + y e_2$

eg $V = \{(x, y, z) \in \mathbb{R}^3 : x - y + z = 0\} \subseteq \mathbb{R}^3$

$v_1 = (1, 0, -1)$, $v_2 = (0, 1, 1) \in V$ not in same/opposite direction

$E = \{v_1, v_2\}$ is a basis of V



For any $\vec{v} \in V$, there exist a unique α, β such that $\vec{v} = \alpha v_1 + \beta v_2$

For example $v = (3, -1, -4) \in V$, $v = 3v_1 + (-1)v_2$

Theorem 1.3.33. The following conditions for $n \times n$ matrix A are equivalent.

1. $\det(A) \neq 0$
2. A is invertible, that is, the inverse A^{-1} of A exists.
3. For any n column vector $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for $\mathbf{x}$.
4. The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has no nontrivial solution, that is, solution for which $\mathbf{x} \neq \mathbf{0}$.
5. The column vectors of A constitute a basis for $\mathbb{R}^n$.

eg 1 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\det A = -2 \neq 0$ A is invertible

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has only zero solution

$\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ is
a basis for $\mathbb{R}^2$

eg 2 $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\det B = 0$ B is not invertible

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has non-trivial solution
for example, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ is
not a basis for $\mathbb{R}^2$

1.7 Some transcendental functions

1. Exponential function:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \text{ for } x \in \mathbb{R}$$

2. Trigonometric functions: *There are 6 trigonometric functions which are defined as follows.*

$$\text{Cosine:} \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \text{ for } x \in \mathbb{R}$$

$$\text{Sine:} \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \text{ for } x \in \mathbb{R}$$

$$\text{Tangent:} \quad \tan x = \frac{\sin x}{\cos x} \text{ for } x \neq \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}$$

$$\text{Cotangent:} \quad \cot x = \frac{\cos x}{\sin x} \text{ for } x \neq k\pi, \quad k \in \mathbb{Z}$$

$$\text{Secant:} \quad \sec x = \frac{1}{\cos x} \text{ for } x \neq \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}$$

$$\text{Cosecant:} \quad \csc x = \frac{1}{\sin x} \text{ for } x \neq k\pi, \quad k \in \mathbb{Z}$$

3. **Hyperbolic functions:** *There are 6 hyperbolic functions which are defined as follows.*

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \text{ for } x \in \mathbb{R}$$

Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \text{ for } x \in \mathbb{R}$$

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} \text{ for } x \in \mathbb{R}$$

Hyperbolic cotangent:

$$\coth x = \frac{\cosh x}{\sinh x} \text{ for } x \neq 0$$

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} \text{ for } x \in \mathbb{R}$$

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} \text{ for } x \neq 0$$

Theorem 1.7.2. *The exponential function satisfies*

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$$

for any $x \in \mathbb{R}$.

Definition 1.7.3 (Logarithmic function). *The logarithmic function is the function $\ln : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined for $x > 0$ by*

$$y = \ln x \text{ if } e^y = x.$$

In other words, $\ln x$ is the inverse function of the exponential function.

Proposition 1.7.4 (Identities for transcendental functions).

1. *Exponential function:*

$$(a) \quad e^{x+y} = e^x e^y$$

$$(b) \quad e^{x-y} = \frac{e^x}{e^y}$$

$$(c) \quad e^{kx} = (e^x)^k \text{ for } k \in \mathbb{Z}$$

2. *Logarithmic function:*

$$(a) \quad \ln(xy) = \ln x + \ln y$$

$$(b) \quad \ln \frac{x}{y} = \ln x - \ln y$$

$$(c) \quad \ln(x^k) = k \ln x \text{ for } k \in \mathbb{Z}$$

3. *Trigonometric identities:*

$$(a) \quad \cos^2 x + \sin^2 x = 1; \quad \sec^2 x - \tan^2 x = 1; \quad \csc^2 x - \cot^2 x = 1$$

$$(b) \quad \cos(-x) = \cos x; \quad \sin(-x) = -\sin x; \quad \tan(-x) = -\tan x$$

$$(c) \quad \cos(x+y) = \cos x \cos y - \sin x \sin y;$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y;$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$(d) \quad \cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x;$$

$$\sin 2x = 2 \sin x \cos x;$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

4. *Hyperbolic identities:*

$$(a) \quad \cosh^2 x - \sinh^2 x = 1; \quad \operatorname{sech}^2 x + \tanh^2 x = 1; \quad \coth^2 x - \operatorname{csch}^2 x = 1$$

$$(b) \quad \cosh(-x) = \cosh x; \quad \sinh(-x) = -\sinh x; \quad \tanh(-x) = -\tanh x$$

$$(c) \quad \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y;$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y;$$

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

$$(d) \quad \cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x;$$

$$\sinh 2x = 2 \sinh x \cosh x;$$

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

Proposition 1.7.5 (Derivatives of transcendental functions).

1. *Exponential function:*

$$\frac{d}{dx}e^x = e^x$$

2. *Logarithmic function:*

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

3. *Trigonometric functions:*

$$\begin{aligned}\frac{d}{dx} \cos x &= -\sin x; & \frac{d}{dx} \sin x &= \cos x; \\ \frac{d}{dx} \tan x &= \sec^2 x; & \frac{d}{dx} \cot x &= -\csc^2 x; \\ \frac{d}{dx} \sec x &= \sec x \tan x; & \frac{d}{dx} \csc x &= -\csc x \cot x\end{aligned}$$

4. *Inverse trigonometric functions*^[7]:

$$\begin{aligned}\frac{d}{dx} \cos^{-1} x &= -\frac{1}{\sqrt{1-x^2}}; \\ \frac{d}{dx} \sin^{-1} x &= \frac{1}{\sqrt{1-x^2}}; \\ \frac{d}{dx} \tan^{-1} x &= \frac{1}{1+x^2}\end{aligned}$$

5. *Hyperbolic functions:*

$$\begin{aligned}\frac{d}{dx} \cosh x &= \sinh x; & \frac{d}{dx} \sinh x &= \cosh x; \\ \frac{d}{dx} \tanh x &= \operatorname{sech}^2 x; & \frac{d}{dx} \coth x &= -\operatorname{csch}^2 x; \\ \frac{d}{dx} \operatorname{sech} x &= -\operatorname{sech} x \tanh x; & \frac{d}{dx} \operatorname{csch} x &= -\operatorname{csch} x \coth x\end{aligned}$$

6. *Inverse hyperbolic functions*^[8]:

$$\begin{aligned}\frac{d}{dx} \cosh^{-1} x &= \frac{1}{\sqrt{x^2-1}}; \\ \frac{d}{dx} \sinh^{-1} x &= \frac{1}{\sqrt{x^2+1}}; \\ \frac{d}{dx} \tanh^{-1} x &= \frac{1}{1-x^2}\end{aligned}$$

Proposition 1.7.6 (Integrals of transcendental functions).

1. *Exponential function:*

$$\int e^x dx = e^x + C$$

2. *Logarithmic function:*

$$\int \frac{1}{x} dx = \ln |x| + C$$

3. *Trigonometric functions:*

$$\int \cos x dx = \sin x + C; \quad \int \sin x dx = -\cos x + C;$$

$$\int \tan x dx = \ln \sec x; \quad \int \cot x = \ln \sin x + C;$$

$$\int \sec x dx = \ln |\sec x + \tan x| + C; \quad \int \csc x dx = \ln |\csc x - \cot x| + C$$

4. *Hyperbolic functions:*

$$\int \cosh x dx = \sinh x + C; \quad \int \sinh x dx = \cosh x + C;$$

$$\int \tanh x dx = \ln \cosh x; \quad \int \coth x = \ln \sinh x + C;$$

$$\int \operatorname{sech} x dx = \tan^{-1} \sinh x + C; \quad \int \operatorname{csch} x dx = \ln |\operatorname{csch} x - \coth x| + C$$

2 Curves

2.1 Regular parametrized curves

Definition 2.1.1 (Regular parametrized curves). A **regular parametrized curve** is a differentiable function $\mathbf{r} : (a, b) \rightarrow \mathbb{R}^n$, $n = 2$ or 3 , such that $\mathbf{r}'(t) \neq \mathbf{0}$ for any $t \in (a, b)$.

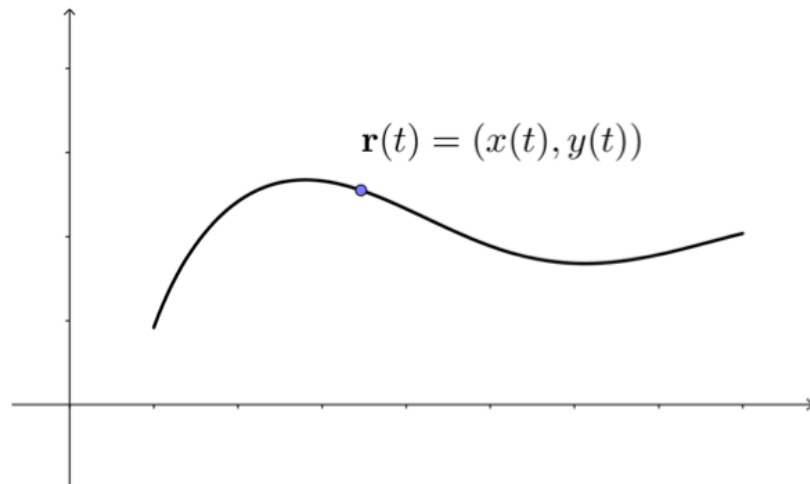
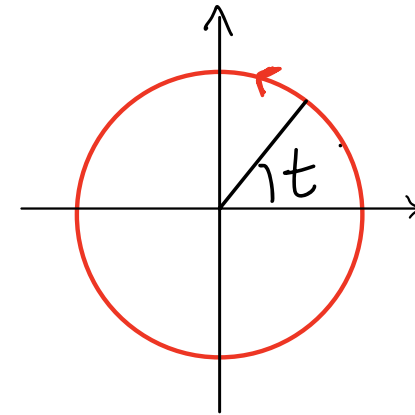


Figure 1: Regular parametrized curve

Rmk • t is called parameter

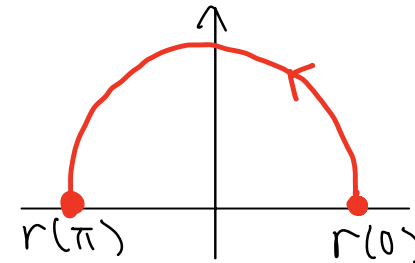
- regular means $\mathbf{r}'(t) \neq \vec{0}$
- We will also consider curve defined on closed interval and unbounded interval

- $\mathbf{r}(t) = (\cos t, \sin t) \quad t \in (-\infty, \infty)$

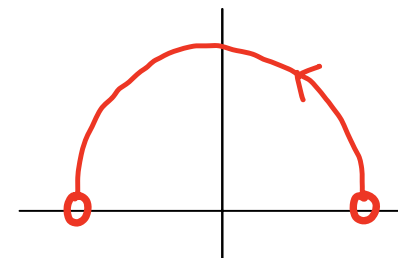


loop around
infinity time

- $\mathbf{r}(t) = (\cos t, \sin t), \quad 0 \leq t \leq \pi$



- $\mathbf{r}(t) = (\cos t, \sin t), \quad 0 < t < \pi$



1. *Straight line:* Let (x_0, y_0) and (x_1, y_1) be two points on $\mathbb{R}^2$. The function

$$\mathbf{r}(t) = ((1-t)x_0 + tx_1, (1-t)y_0 + ty_1), \text{ for } 0 < t < 1$$

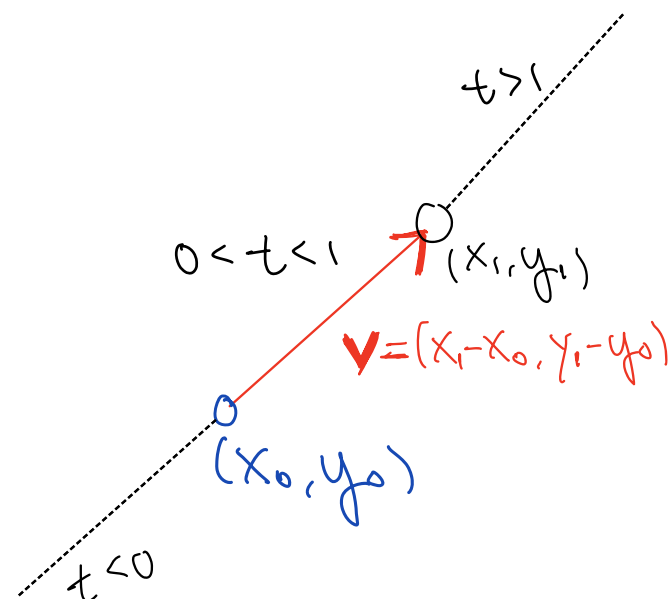
$$\vec{r}(t) = (x_0 - tx_0 + tx_1, y_0 - ty_0 + ty_1)$$

$$= (x_0, y_0) + (t(x_1 - x_0), t(y_1 - y_0))$$

$$= \underbrace{(x_0, y_0)}_{\text{"Starting point"}} + t \underbrace{(x_1 - x_0, y_1 - y_0)}_{\text{vector from } (x_0, y_0) \text{ to } (x_1, y_1)}$$

"Starting point"

vector from (x_0, y_0)
to (x_1, y_1)



2. *Circle:* Let $r > 0$ be a positive real number. The function

$$\mathbf{r}(\theta) = (r \cos \theta, r \sin \theta), \text{ for } 0 < \theta < 2\pi$$

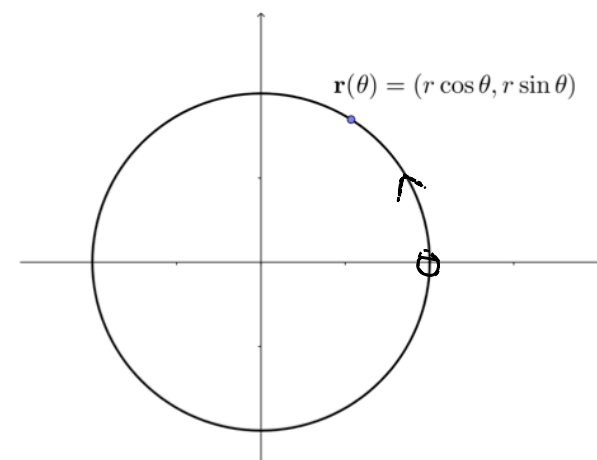


Figure 3: Circle

3. Cycloid: The function

$$\mathbf{r}(\theta) = (\theta - \sin \theta, 1 - \cos \theta), \text{ for } 0 < \theta < 2\pi$$

$$= \underbrace{(\theta, 1)}_{\mathbf{r}_1} + \underbrace{(-\sin \theta, -\cos \theta)}_{\mathbf{r}_2}$$

$$\mathbf{r}_2(\theta)$$

$$= (-\sin \theta, -\cos \theta)$$

$$= \left(\cos\left(\frac{3\pi}{2} - \theta\right), \sin\left(\frac{3\pi}{2} - \theta\right) \right)$$

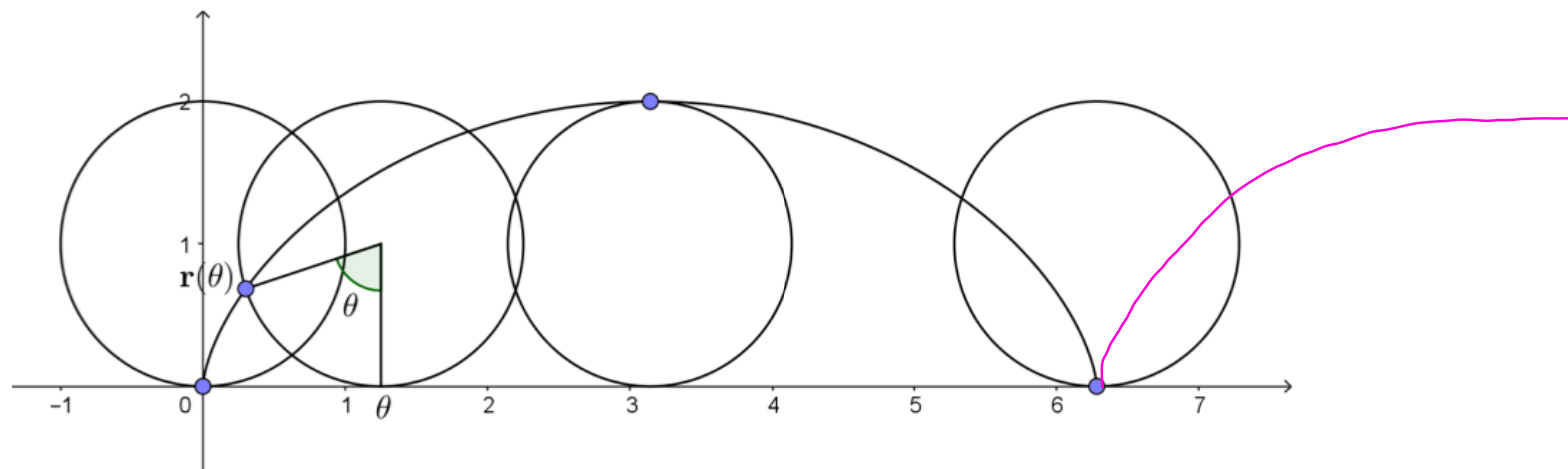
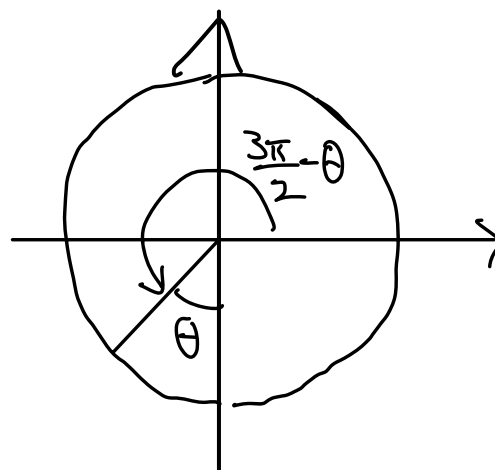
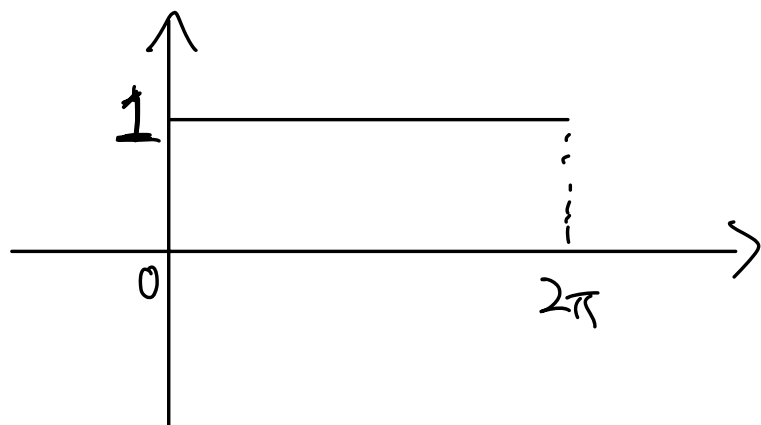


Figure 4: Cycloid

Q Consider $r(\theta) = (\theta - \sin\theta, 1 - \cos\theta)$ $0 < \theta < 4\pi$

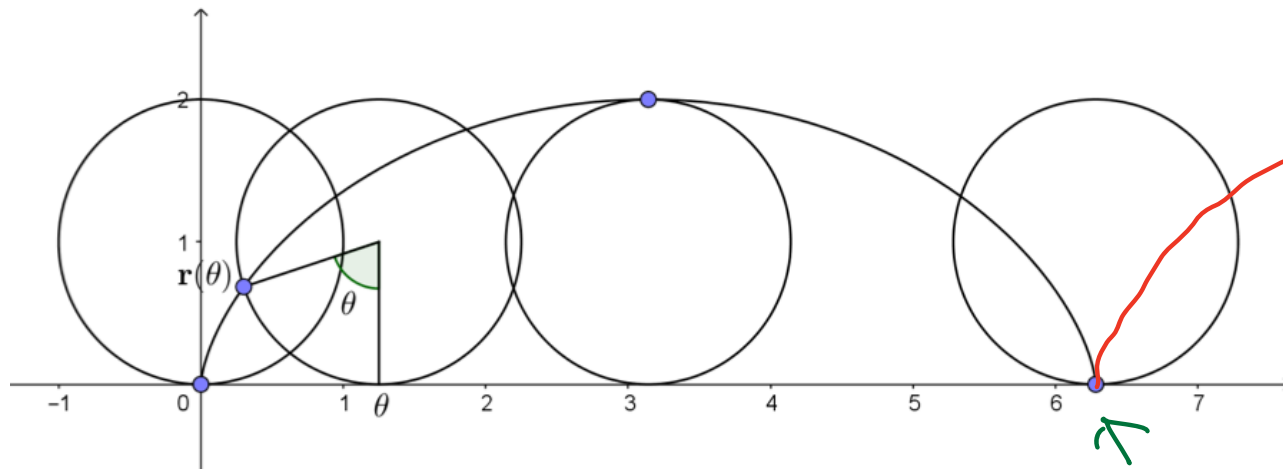


Figure 4: Cycloid

sharp corner?
differentiable?

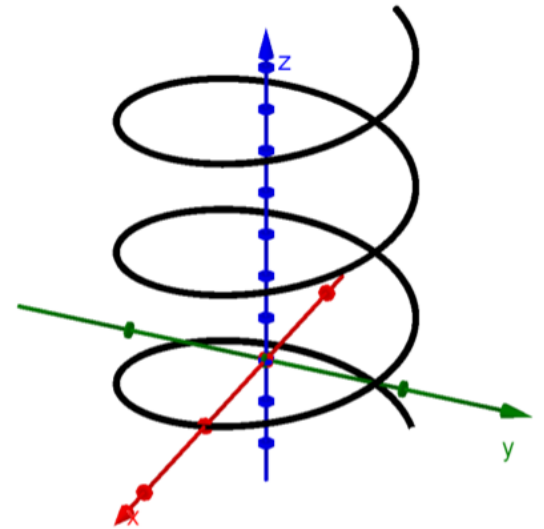
Note $r'(\theta) = (1 - \cos\theta, \sin\theta)$

$r'(2\pi) = (1 - 1, 0) = (0, 0) \leftarrow$ not regular

4. Helix: The function $\mathbf{r}(\theta) = (a \cos \theta, a \sin \theta, b\theta)$

$$\mathbf{r}(\theta) = (a \cos \theta, a \sin \theta, b\theta), \text{ for } \theta \in \mathbb{R}$$

defines a curve which is called a **helix**.



Example 2.1.3. Let $\mathbf{r}(t) = (t^2, t^3)$. Then $\mathbf{r}'(t) = (2t, 3t^2)$ and $\mathbf{r}'(0) = (0, 0)$. Therefore $\mathbf{r}(t)$ is not regular at $t = 0$.

$$\begin{cases} x = t^2 \\ y = t^3 \end{cases} \Rightarrow \begin{aligned} x^3 &= y^2 \\ y &= \pm x^{3/2} \end{aligned}$$

