MMAT5390: Mathematical Image Processing

Midterm practice

1. Recall that an image transformation $\mathcal{O} : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ is said to be separable if there exist matrices $A \in M_{n\times n}(\mathbb{R})$ and $B \in M_{n\times m}(\mathbb{R})$ such that $\mathcal{O}(f) = AfB$ for any $f \in M_{n\times n}(\mathbb{R})$.

   Here $\mathcal{O} : M_{3\times 3}(\mathbb{R}) \to M_{3\times 3}(\mathbb{R})$ is an image transformation and the transformation matrix of its PSF is

$$H = \begin{pmatrix} 2 & 4 & 6 & 2 & 4 & 6 & 0 & 0 & 0 \\ 8 & 10 & 0 & 8 & 10 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 & 3 & 6 & 9 \\ 4 & 5 & 0 & 0 & 0 & 0 & 12 & 15 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 18 & 0 & 0 \\ 0 & 0 & 0 & 4 & 8 & 12 & 1 & 2 & 3 \\ 0 & 0 & 0 & 16 & 20 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 24 & 0 & 0 & 6 & 0 & 0 \end{pmatrix}.$$

   Please determine if $\mathcal{O}$ is separable. If yes, please find the corresponding matrices $A \in M_{3\times 3}(\mathbb{R})$ and $B \in M_{3\times 3}(\mathbb{R})$.

   Solution: Let $A = (a_{ij})_{1\le i,j\le 3}$, $B = (b_{ij})_{1\le i,j\le 3}$ and $g = \mathcal{O}(f) \in M_{3\times 3}(\mathbb{R})$, then we have

$$g_{\alpha,\beta} = \sum_{x=1}^{3} a_{\alpha x}\Big(\sum_{y=1}^{3} f(x,y) b_{y\beta}\Big) = \sum_{x=1}^{3}\sum_{y=1}^{3} a_{\alpha x} b_{y\beta} f(x,y),$$

   Which means $h^{\alpha,\beta}(x,y) = a_{\alpha x} b_{y\beta}$. Hence the transformation matrix

$$H = \begin{pmatrix} a_{11}b_{11} & a_{12}b_{11} & a_{13}b_{11} & a_{11}b_{21} & a_{12}b_{21} & a_{13}b_{21} & a_{11}b_{31} & a_{12}b_{31} & a_{13}b_{31} \\ a_{21}b_{11} & a_{22}b_{11} & a_{23}b_{11} & a_{21}b_{21} & a_{22}b_{21} & a_{23}b_{21} & a_{21}b_{31} & a_{22}b_{31} & a_{23}b_{31} \\ a_{31}b_{11} & a_{32}b_{11} & a_{33}b_{11} & a_{31}b_{21} & a_{32}b_{21} & a_{33}b_{21} & a_{31}b_{31} & a_{32}b_{31} & a_{33}b_{31} \\ a_{11}b_{12} & a_{12}b_{12} & a_{13}b_{12} & a_{11}b_{22} & a_{12}b_{22} & a_{13}b_{22} & a_{11}b_{32} & a_{12}b_{32} & a_{13}b_{32} \\ a_{21}b_{12} & a_{22}b_{12} & a_{23}b_{12} & a_{21}b_{22} & a_{22}b_{22} & a_{23}b_{22} & a_{21}b_{32} & a_{22}b_{32} & a_{23}b_{32} \\ a_{31}b_{12} & a_{32}b_{12} & a_{33}b_{12} & a_{31}b_{22} & a_{32}b_{22} & a_{33}b_{22} & a_{31}b_{32} & a_{32}b_{32} & a_{33}b_{32} \\ a_{11}b_{13} & a_{12}b_{13} & a_{13}b_{13} & a_{11}b_{23} & a_{12}b_{23} & a_{13}b_{23} & a_{11}b_{33} & a_{12}b_{33} & a_{13}b_{33} \\ a_{21}b_{13} & a_{22}b_{13} & a_{23}b_{13} & a_{21}b_{23} & a_{22}b_{23} & a_{23}b_{23} & a_{21}b_{33} & a_{22}b_{33} & a_{23}b_{33} \\ a_{31}b_{13} & a_{32}b_{13} & a_{33}b_{13} & a_{31}b_{23} & a_{32}b_{23} & a_{33}b_{23} & a_{31}b_{33} & a_{32}b_{33} & a_{33}b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11}A & b_{21}A & b_{31}A \\ b_{12}A & b_{22}A & b_{32}A \\ b_{13}A & b_{23}A & b_{33}A \end{pmatrix} = B^T \otimes A.$$

   At the same time, it's easy to notice that H is the kronecker product of two 3×3 matrices; explicitly, let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 4 \\ 0 & 3 & 1 \end{pmatrix}$ then $H = B^T \otimes A$. So the image transformation $\mathcal{O}$ is separable and the required A, B are given above.

2. A matrix $H \in M_{n^2\times n^2}(\mathbb{R})$ is called block-circulant if it has the form

$$H = \begin{pmatrix} H_1 & H_n & \cdots & H_2 \\ H_2 & H_1 & \cdots & H_3 \\ \vdots & \vdots & \ddots & \vdots \\ H_n & H_{n-1} & \cdots & H_1 \end{pmatrix},$$

where $H_i \in M_{n\times n}(\mathbb{R})$ for $i = 1, \cdots, n$. Given matrix $k, f \in M_{n\times n}(\mathbb{R})$, let the image transformation $\mathcal{O}(f) = k * f$, please prove that the transformation matrix H of $\mathcal{O}$ is block-circulant.

Solution: Let $g = \mathcal{O}(f) \in M_{n\times n}(\mathbb{R})$. Then for any $1 \le \alpha, \beta \le n$, we have

$$g_{\alpha,\beta} = \sum_{x=1}^{n}\sum_{y=1}^{n} k_{\alpha-x,\beta-y} f_{x,y}$$

which means $h^{\alpha,\beta}(x,y) = k_{\alpha-x,\beta-y}$. Hence the transformation matrix

$$H = \begin{pmatrix} k_{n,n} & \cdots & k_{1,n} & k_{n,n-1} & \cdots & k_{1,n-1} & & k_{n,1} & \cdots & k_{1,1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ k_{n-1,n} & \cdots & k_{n,n} & k_{n-1,n-1} & \cdots & k_{n,n-1} & & k_{n-1,1} & \cdots & k_{n,1} \\ k_{n,1} & \cdots & k_{1,1} & k_{n,n} & \cdots & k_{1,n} & & k_{n,2} & \cdots & k_{1,2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ k_{n-1,1} & \cdots & k_{n,1} & k_{n-1,n} & \cdots & k_{n,n} & & k_{n-1,2} & \cdots & k_{n,2} \\ & \vdots & & & \vdots & & \ddots & & \vdots & \\ k_{n,n-1} & \cdots & k_{1,n-1} & k_{n,n-2} & \cdots & k_{1,n-2} & & k_{n,n} & \cdots & k_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ k_{n-1,n-1} & \cdots & k_{n,n-1} & k_{n-1,n-2} & \cdots & k_{n,n-2} & & k_{n-1,n} & \cdots & k_{n,n} \end{pmatrix}$$

$$= \begin{pmatrix} H_1 & H_n & \cdots & H_2 \\ H_2 & H_1 & \cdots & H_3 \\ \vdots & \vdots & \ddots & \vdots \\ H_n & H_{n-1} & \cdots & H_1 \end{pmatrix}$$

where $H_k = \begin{pmatrix} k_{n,k-1} & k_{n-1,k-1} & \cdots & k_{2,k-1} & k_{1,k-1} \\ k_{1,k-1} & k_{n,k-1} & \cdots & k_{3,k-1} & k_{2,k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{n-2,k-1} & k_{n-3,k-1} & \cdots & k_{n,k-1} & k_{n-1,k-1} \\ k_{n-1,k-1} & k_{n-2,k-1} & \cdots & k_{1,k-1} & k_{n,k-1} \end{pmatrix}$. Hence the transformation matrix H of $\mathcal{O}$ is block-circulant.

3. Let $H = \begin{pmatrix} r & 2r & u & 2u \\ 3r & r & 3v & v \\ 3 & 6 & s & 2s \\ 9 & 3 & 3s & s \end{pmatrix}$ be the transformation matrix corresponding to an image transformation $\mathcal{O} : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$, where r, s, u, v are all non-zero real numbers. Prove that $\mathcal{O}$ is separable and if and only if $u = v$. Please explain your answer with details.

Solution: $\Rightarrow$: If $u = v$, we have $H = \begin{pmatrix} r & u \\ 3 & s \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, then

$$\mathcal{O}(f) = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} f \begin{pmatrix} r & 3 \\ u & s \end{pmatrix}$$

for all $f \in M_{2\times 2}(\mathbb{R})$. Hence $\mathcal{O}$ is separable.

$\Leftarrow$: If $\mathcal{O}$ is separable, then there exist $A, B \in M_{2\times 2}(\mathbb{R})$ such that $\mathcal{O}(f) = AfB$ for all $f \in M_{2\times 2}(\mathbb{R})$. So the transformation matrix H of $\mathcal{O}$ is given by

$$H = \begin{pmatrix} b_{11}a_{11} & b_{11}a_{12} & b_{21}a_{11} & b_{21}a_{12} \\ b_{11}a_{21} & b_{11}a_{22} & b_{21}a_{21} & b_{21}a_{22} \\ b_{12}a_{11} & b_{12}a_{12} & b_{22}a_{11} & b_{22}a_{12} \\ b_{12}a_{21} & b_{12}a_{22} & b_{22}a_{21} & b_{22}a_{22} \end{pmatrix}$$

where a_{ij} and b_{ij} are the entries of A and B respectively and $1 \le i, j \le 2$. Since we have $b_{12}a_{11} = b_{12}a_{22} = 3$, then $a_{11} = a_{22} \neq 0$ and so $u = b_{21}a_{11} = b_{21}a_{22} = v$.

4. Let $H = \begin{pmatrix} 2 & 4 & a & 6 \\ 4 & 2 & 6 & 1 \\ 1 & 6 & 2 & 4 \\ c & 1 & 4 & b \end{pmatrix}$ be the transformation matrix corresponding to an image transformation $\mathcal{O} : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$, where a, b, c are all non-zero real numbers. Please determine a, b, c such that $\mathcal{O}$ is an image transformation defined by convolution.

Solution: We have proved the transformation matrix H of $\mathcal{O}$ is block-circulant if $\mathcal{O}$ is an image transformation defined by convolution. So $a = 1, b = 2, c = 6$.

5. Let $f = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 4 & 0 & 2 \end{pmatrix}$.

   (a) Compute an SVD of f.

   (b) Express f as a linear combination of its elementary images.

Solution:

(a) $ff^T = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}$.

For $\lambda = 20$:

$$\left[\begin{array}{cc|c} -10 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

which gives unit eigenvector $\vec{u}_1 = (0, 1)^T$.

For $\lambda = 10$:

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -10 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

which gives unit eigenvector $\vec{u}_2 = (1, 0)^T$.

Then $\vec{v}_1 = \frac{f^T\vec{u}_1}{\sigma_1} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}(0, 2, 0, 1)^T$, and

$\vec{v}_2 = \frac{f^T\vec{u}_2}{\sigma_2} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{10}}(1, 0, 3, 0)^T$.

$f^T f = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 16 & 0 & 8 \\ 3 & 0 & 9 & 0 \\ 0 & 8 & 0 & 4 \end{pmatrix}$.

For $f^T f \vec{v} = 0$,

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 16 & 0 & 8 & 0 \\ 3 & 0 & 9 & 0 & 0 \\ 0 & 8 & 0 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

which gives orthonormal eigenvectors $\vec{v}_3 = \frac{1}{\sqrt{10}}(-3, 0, 1, 0)^T$ and $\vec{v}_4 = \frac{1}{\sqrt{5}}(0, 1, 0, -2)^T$.

Hence an SVD of f is $f = U\Sigma V^T$, where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \Sigma = \begin{pmatrix} 2\sqrt{5} & 0 & 0 & 0 \\ 0 & \sqrt{10} & 0 & 0 \end{pmatrix} \text{ and } V = \frac{1}{\sqrt{10}} \begin{pmatrix} 0 & 1 & -3 & 0 \\ 2\sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & 3 & 1 & 0 \\ \sqrt{2} & 0 & 0 & -2\sqrt{2} \end{pmatrix}.$$

(b) The eigenimages are given by

$\vec{u}_1\vec{v}_1^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \end{pmatrix}$ and

$\vec{u}_2\vec{v}_2^T = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{10}}{10} & 0 & \frac{3\sqrt{10}}{10} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Hence $f = 2\sqrt{5}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \end{pmatrix} + \sqrt{10}\begin{pmatrix} \frac{\sqrt{10}}{10} & 0 & \frac{3\sqrt{10}}{10} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

6. Let $f = \begin{pmatrix} 5 & 4 & 6 & 6 \\ 6 & 1 & 6 & 3 \\ 1 & 2 & 1 & 5 \\ 6 & 4 & 6 & 1 \end{pmatrix}$.

   (a) Compute the Haar transform f_{Haar} of f.

   (b) Suppose there is only enough capacity to store 10 pixel values of f_{Haar}. Choose 10 entries to keep such that the reconstucted image differs as little as possible in Frobenius norm with the original image, and compute the reconstructed image.

Solution:

(a) $\tilde{H} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$.

$$
\begin{aligned}
f_{\text{Haar}} &= \tilde{H} f \tilde{H}^T \\
&= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 5 & 4 & 6 & 6 \\ 6 & 1 & 6 & 3 \\ 1 & 2 & 1 & 5 \\ 6 & 4 & 6 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 18 & 11 & 19 & 15 \\ 4 & -1 & 5 & 3 \\ -\sqrt{2} & 3\sqrt{2} & 0 & 3\sqrt{2} \\ -5\sqrt{2} & -2\sqrt{2} & -5\sqrt{2} & 4\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 63 & -5 & 7\sqrt{2} & 4\sqrt{2} \\ 11 & -5 & 5\sqrt{2} & 2\sqrt{2} \\ 5\sqrt{2} & -\sqrt{2} & -8 & -6 \\ -8\sqrt{2} & -6\sqrt{2} & -6 & -18 \end{pmatrix} = \begin{pmatrix} \frac{63}{4} & -\frac{5}{4} & \frac{7\sqrt{2}}{4} & \sqrt{2} \\ \frac{11}{4} & -\frac{5}{4} & \frac{5\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -2 & -\frac{3}{2} \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & -\frac{3}{2} & -\frac{9}{2} \end{pmatrix}.
\end{aligned}
$$

(b) Since $\tilde{H}$ is unitary, for any $g \in M_{4\times 4}(\mathbb{R})$,

$$
\|\tilde{H}^T g \tilde{H} - f\|_F = \|\tilde{H}^T (g - f_{\text{Haar}})\tilde{H}\|_F = \|g - f_{\text{Haar}}\|_F.
$$

Hence one should choose to discard the entries with smaller absolute values so as to minimize the Frobenius norm of the difference.

Hence the matrix that should be kept is either $f'_{\text{Haar}} = \begin{pmatrix} \frac{63}{4} & 0 & \frac{7\sqrt{2}}{4} & 0 \\ \frac{11}{4} & 0 & \frac{5\sqrt{2}}{4} & 0 \\ \frac{5\sqrt{2}}{4} & 0 & -2 & 0 \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & -\frac{3}{2} & -\frac{9}{2} \end{pmatrix}$,

whose reconstructed image is given by $\tilde{H}^T f'_{\text{Haar}} \tilde{H}$

$$
\begin{aligned}
&= \frac{1}{16}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 63 & 0 & 7\sqrt{2} & 0 \\ 11 & 0 & 5\sqrt{2} & 0 \\ 5\sqrt{2} & 0 & -8 & 0 \\ -8\sqrt{2} & -6\sqrt{2} & -6 & -18 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix} 84 & 0 & 4\sqrt{2} & 0 \\ 64 & 0 & 20\sqrt{2} & 0 \\ 36 & -12 & -4\sqrt{2} & -18\sqrt{2} \\ 68 & 12 & 8\sqrt{2} & 18\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix} 92 & 76 & 84 & 84 \\ 104 & 24 & 64 & 64 \\ 16 & 32 & 12 & 84 \\ 96 & 64 & 92 & 20 \end{pmatrix} = \begin{pmatrix} \frac{23}{4} & \frac{19}{4} & \frac{21}{4} & \frac{21}{4} \\ \frac{13}{2} & \frac{3}{2} & 4 & 4 \\ 1 & 2 & \frac{3}{4} & \frac{21}{4} \\ 6 & 4 & \frac{23}{4} & \frac{5}{4} \end{pmatrix};
\end{aligned}
$$

or keep $f''_{\text{Haar}} = \begin{pmatrix} \frac{63}{4} & 0 & \frac{7\sqrt{2}}{4} & 0 \\ \frac{11}{4} & 0 & \frac{5\sqrt{2}}{4} & 0 \\ \frac{5\sqrt{2}}{4} & 0 & -2 & -\frac{3}{2} \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & 0 & -\frac{9}{2} \end{pmatrix}$,

whose reconstructed image is given by $\tilde{H}^T f''_{\text{Haar}} \tilde{H}$

$$
\begin{aligned}
&= \frac{1}{16} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 63 & 0 & 7\sqrt{2} & 0 \\ 11 & 0 & 5\sqrt{2} & 0 \\ 5\sqrt{2} & 0 & -8 & -6 \\ -8\sqrt{2} & -6\sqrt{2} & 0 & -18 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{16} \begin{pmatrix} 84 & 0 & 4\sqrt{2} & -6\sqrt{2} \\ 64 & 0 & 20\sqrt{2} & 6\sqrt{2} \\ 36 & -12 & 2\sqrt{2} & -18\sqrt{2} \\ 68 & 12 & 2\sqrt{2} & 18\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{16} \begin{pmatrix} 92 & 76 & 72 & 96 \\ 104 & 24 & 76 & 52 \\ 28 & 20 & 12 & 84 \\ 84 & 76 & 92 & 20 \end{pmatrix} = \begin{pmatrix} \frac{23}{4} & \frac{19}{4} & \frac{9}{2} & 6 \\ \frac{13}{2} & \frac{3}{2} & \frac{19}{4} & \frac{13}{4} \\ \frac{7}{4} & \frac{5}{4} & \frac{3}{4} & \frac{21}{4} \\ \frac{21}{4} & \frac{19}{4} & \frac{23}{4} & \frac{5}{4} \end{pmatrix}.
\end{aligned}
$$

7. Let $H_n(t)$ be the n^{th} Haar function, where $n \in \mathbb{N} \cup \{0\}$.

   (a) Write down the definition of $H_n(t)$.

   (b) Write down the Haar transform matrix $\tilde{H}$ for 4×4 images.

   (c) Suppose $A = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 4 & 5 & 5 & 6 \end{pmatrix}$. Compute the Haar transform A_{Haar} of A, and compute the reconstructed image $\tilde{A}$ after setting the largest entry of A_{Haar} to 0.

Solution:

(a) $H_0(t) = \mathbf{1}_{[0,1)}$, and for any $p \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{Z} \cap [0, 2^p - 1]$,

$$H_{2^p+n}(t) = 2^{\frac{p}{2}} \left(\mathbf{1}_{[\frac{n}{2^p}, \frac{n+0.5}{2^p})} - \mathbf{1}_{[\frac{n+0.5}{2^p}, \frac{n+1}{2^p})} \right).$$

(b) $\tilde{H} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$.

(c)

$$
\begin{aligned}
A_{\text{Haar}} &= \tilde{H} A \tilde{H}^T \\
&= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 4 & 5 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 8 & 12 & 12 & 16 \\ -4 & -4 & -4 & -4 \\ -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} \\ -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 12 & -2 & -\sqrt{2} & -\sqrt{2} \\ -4 & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Then the modified Haar transform A'_{Haar} is $\begin{pmatrix} 0 & -2 & -\sqrt{2} & -\sqrt{2} \\ -4 & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \end{pmatrix}$, and thus:

$$
\begin{aligned}
\tilde{A} &= \tilde{H}^T A'_{\text{Haar}} \tilde{H} \\
&= \frac{1}{4} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & -2 & -\sqrt{2} & -\sqrt{2} \\ -4 & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} -8 & -2 & -\sqrt{2} & -\sqrt{2} \\ 0 & -2 & -\sqrt{2} & -\sqrt{2} \\ 0 & -2 & -\sqrt{2} & -\sqrt{2} \\ 8 & -2 & -\sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} -3 & -2 & -2 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 3 \end{pmatrix}.
\end{aligned}
$$

8. Suppose the definition of the DFT on $N \times N$ images is changed to

$$\hat{f}(m,n) = DFT(f)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k,l) e^{2\pi j \frac{mk+nl}{N}}.$$

(a) Does there exist a matrix U such that $\hat{f} = UfU$ for an $N \times N$ image f? If yes, derive U and check if it is unitary.

(b) Show that the inverse DFT (iDFT) is defined by

$$f(p,q) = iDFT(\hat{f})(p,q) = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \hat{f}(m,n) e^{-2\pi j \frac{pm+qn}{N}}.$$

Solution:

(a) The matrix U used to calculate the DFT of an $N \times N$ matrix is given by

$$U = (U(x,\alpha))_{0 \le x, \alpha \le n}, \text{ where } U(x,\alpha) = \frac{1}{\sqrt{N}} e^{2\pi j \frac{\alpha x}{N}}.$$

To check that U is unitary, we first denote the column of U indexed by α by $\vec{u}_\alpha$. Then,

i. For any $0 \le \alpha \le N-1$,

$$
\begin{aligned}
\langle \vec{u}_\alpha, \vec{u}_\alpha \rangle &= \sum_{x=0}^{N-1} U(x,\alpha) \overline{U(x,\alpha)} \\
&= \sum_{x=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi j \frac{x\alpha}{N}} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{x\alpha}{N}} \\
&= N \cdot \frac{1}{N} = 1.
\end{aligned}
$$

ii. For any $0 \le \alpha_1, \alpha_2 \le N-1$ such that $\alpha_1 \ne \alpha_2$,

$$
\begin{aligned}
\langle \vec{u}_{\alpha_1}, \vec{u}_{\alpha_2} \rangle &= \sum_{x=0}^{N-1} U(x,\alpha_1) \overline{U(x,\alpha_2)} \\
&= \sum_{x=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi j \frac{x\alpha_1}{N}} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{x\alpha_2}{N}} \\
&= \frac{1}{N} \sum_{x=0}^{N-1} e^{2\pi j \frac{x(\alpha_1 - \alpha_2)}{N}} \\
&= 0.
\end{aligned}
$$

Hence U is unitary.

(b) For any $0 \le p, q \le N-1$,

$$\begin{aligned}
iDFT(DFT(f))(p,q) &= \frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)e^{2\pi j\frac{m(k-p)+n(l-q)}{N}} \\
&= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)\left[\sum_{m=0}^{N-1} e^{2\pi j\frac{m(k-p)}{N}}\right]\left[\sum_{n=0}^{N-1} e^{2\pi j\frac{n(l-q)}{N}}\right] \\
&= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)\cdot N\mathbf{1}_{N\mathbb{Z}}(k-p)\cdot N\mathbf{1}_{N\mathbb{Z}}(l-q) \\
&= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)\delta(k-p)\delta(l-q) = f(p,q).
\end{aligned}$$

9. Let $f = \begin{pmatrix} 3 & 2 & 4 & 4 \\ 4 & -3 & 4 & 0 \\ -2 & -1 & -2 & 3 \\ 4 & 1 & 4 & -2 \end{pmatrix}$.

(a) Compute the discrete Fourier transform $\hat{f}$ of f.

(b) Compute the image reconstructed from $\hat{f}$ after removing frequencies in 3rd row and 3rd column.

Solution:

(a) $U = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$.

$$\begin{aligned}
\hat{f} &= UfU \\
&= \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}\begin{pmatrix} 3 & 2 & 4 & 4 \\ 4 & -3 & 4 & 0 \\ -2 & -1 & -2 & 3 \\ 4 & 1 & 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix} 9 & -1 & 10 & 5 \\ 5 & 3+4j & 6 & 1-2j \\ -7 & 3 & -6 & 9 \\ 5 & 3-4j & 6 & 1+2j \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix} 23 & -1+6j & 15 & -1-6j \\ 15+2j & 5-2j & 7-2j & -7+2j \\ -1 & -1+6j & -25 & -1-6j \\ 15-2j & -7-2j & 7+2j & 5+2j \end{pmatrix}.
\end{aligned}$$

(b) The submatrix of $\hat{f}$ is

$$\hat{f}' = \frac{1}{16}\begin{pmatrix} 23 & -1+6j & 0 & -1-6j \\ 15+2j & 5-2j & 0 & -7+2j \\ 0 & 0 & 0 & 0 \\ 15-2j & -7-2j & 0 & 5+2j \end{pmatrix},$$

whose reconstructed image is

$$(4\bar{U})\hat{f}'(4\bar{U}) = \frac{1}{16}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{pmatrix}\begin{pmatrix}23 & -1+6j & 0 & -1-6j\\ 15+2j & 5-2j & 0 & -7+2j\\ 0 & 0 & 0 & 0\\ 15-2j & -7-2j & 0 & 5+2j\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{pmatrix}$$
$$= \frac{1}{16}\begin{pmatrix}47 & 49 & 59 & 57\\ 17 & -17 & 21 & 55\\ -5 & -27 & -9 & 13\\ 25 & 39 & 29 & 15\end{pmatrix}.$$

10. Let $f, g \in M_{M\times N}(\mathbb{R})$ be periodically extended, please prove $\widehat{f * g} = MN\hat{f}\odot\hat{g}$, where $\hat{f}\odot\hat{g}(m,n) = \hat{f}(m,n)\hat{g}(m,n)$.
Solution:
   - Method 1 (directly): Refer to DFT of convolution of **Further properties of DFT** in Section 2.3.
   - Method 2 (iDFT):

$$\begin{aligned} iDFT(MN\hat{f}\odot\hat{g})(k,l) &= MN\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\hat{f}(m,n)\hat{g}(m,n)e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})}\\ &= \frac{1}{MN}\sum_{m,k',k''=0}^{M-1}\sum_{n,l',l''=0}^{N-1} f(k',l')g(k'',l'')e^{2\pi j(\frac{m(k-k'-k'')}{M}+\frac{n(l-l'-l'')}{N})}\\ &= \sum_{k',k''=0}^{M-1}\sum_{l',l''=0}^{N-1} f(k',l')g(k'',l'')\mathbf{1}_{M\mathbb{Z}}(k-k'-k'')\mathbf{1}_{N\mathbb{Z}}(l-l'-l'')\\ &= \sum_{k',k''=0}^{M-1}\sum_{l',l''=0}^{N-1} f(k',l')g(k'',l'')[\delta(k-k'-k'')+\delta(k-k'-k''+M)]\\ &\quad [\delta(l-l'-l'')+\delta(l-l'-l''+N)]\\ &= \sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1} f(k',l')g(k-k',l-l') = f*g(k,l). \end{aligned}$$

11. Let $f, g \in M_{M\times N}(\mathbb{R})$ be periodically extended, please prove $\widehat{f\odot g} = \hat{f}*\hat{g}$, where $f\odot g(k,l) = f(k,l)g(k,l)$.
Solution:
   - Method 1 (directly):

$$\widehat{f\odot g}(m,n) = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} f(k,l)g(k,l)e^{-2\pi j(\frac{mk}{M}+\frac{nl}{N})},$$

whereas

$$\begin{aligned} \hat{f}*\hat{g}(m,n) &= \sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1}\hat{f}(m',n')\hat{g}(m-m',g-g')\\ &= \frac{1}{M^2N^2}\sum_{m',k,k'=0}^{M-1}\sum_{n',l,l'=0}^{N-1} f(k,l)e^{-2\pi j(\frac{m'k}{M}+\frac{n'l}{N})}g(k',l')e^{-2\pi j(\frac{(m-m')k'}{M}+\frac{(n-n')l'}{N})}\\ &= \frac{1}{M^2N^2}\sum_{m',k,k'=0}^{M-1}\sum_{n'l,l'=0}^{N-1} f(k,l)g(k',l')e^{-2\pi j(\frac{mk'+m'(k-k')}{M}+\frac{nl'+n'(l-l')}{N})}\\ &= \frac{1}{MN}\sum_{k,k'=0}^{M-1}\sum_{l,l'=0}^{N-1} f(k,l)g(k',l')e^{-2\pi j(\frac{mk'}{M}+\frac{nl'}{N})}\delta(k-k')\delta(l-l')\\ &= \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} f(k,l)g(k,l)e^{-2\pi(\frac{mk}{M}+\frac{nl}{N})} = \widehat{f\odot g}(m,n). \end{aligned}$$

- Method 2 (iDFT):

$$\begin{aligned}
iDFT(\hat{f} * \hat{g})(k,l) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \hat{f} * \hat{g}(m,n) e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})} \\
&= \sum_{m,m'=0}^{M-1}\sum_{n,n'=0}^{N-1} \hat{f}(m',n')\hat{g}(m-m',n-n') e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})} \\
&= \frac{1}{M^2N^2} \sum_{m,m',k',k''=0}^{M-1}\sum_{n,n',l',l''=0}^{N-1} f(k',l')g(k'',l'') \\
&\quad e^{2\pi j(\frac{m(k-k'')+m'(k''-k')}{M}+\frac{n(l-l'')+n'(l''-l')}{N})} \\
&= \sum_{k',k''=0}^{M-1}\sum_{l',l''=0}^{N-1} f(k',l')g(k'',l'') \\
&\quad \mathbf{1}_{M\mathbb{Z}}(k-k'')\mathbf{1}_{M\mathbb{Z}}(k'-k'')\mathbf{1}_{N\mathbb{Z}}(l-l'')\mathbf{1}_{N\mathbb{Z}}(l'-l'') \\
&= f(k,l)g(k,l).
\end{aligned}$$

12. Let $f \in M_{N\times N}(\mathbb{R})$ be periodically extended, and let $\tilde{f}(k,l) = f(l,-k)$, please prove $\hat{\tilde{f}} = \tilde{\hat{f}}$.
Solution:

- Method 1 (directly):

$$\begin{aligned}
\hat{\tilde{f}}(m,n) &= \frac{1}{N^2}\sum_{k,l=0}^{N-1} \tilde{f}(k,l) e^{-2\pi j\frac{mk+nl}{N}} \\
&= \frac{1}{N^2}\sum_{k,l=0}^{N-1} f(l,-k) e^{-2\pi j\frac{mk+nl}{N}},
\end{aligned}$$

whereas

$$\begin{aligned}
\tilde{\hat{f}}(m,n) &= \hat{f}(n,-m) \\
&= \frac{1}{N^2}\sum_{k,l=0}^{N-1} f(k,l) e^{-2\pi j\frac{nk-ml}{N}} \\
&= \frac{1}{N^2}\sum_{l'=0}^{N-1}\sum_{k'=1-N}^{0} f(l',-k') e^{-2\pi j\frac{mk'+nl'}{N}} \\
&= \frac{1}{N^2}\sum_{l'=0}^{N-1}\left(f(l',0)e^{-2\pi j\frac{nl'}{N}} + \sum_{k'=1-N}^{-1} f(l',-k') e^{-2\pi j\frac{mk'+nl'}{N}} \right) \\
&= \frac{1}{N^2}\sum_{k',l'=0}^{N-1} f(l',-k') e^{-2\pi j\frac{mk'+nl'}{N}} = \hat{\tilde{f}}(m,n).
\end{aligned}$$

- Method 2 (iDFT):

$$
\begin{aligned}
iDFT(\tilde{\hat{f}})(k,l) &= \sum_{m,n=0}^{N-1} \tilde{\hat{f}}(m,n) e^{2\pi j \frac{mk+nl}{N}} \\
&= \sum_{m,n=0}^{N-1} \hat{f}(n,-m) e^{2\pi j \frac{mk+nl}{N}} \\
&= \sum_{m'=0}^{N-1} \sum_{n'=1-N}^{0} \hat{f}(m',n') e^{2\pi j \frac{-n'k+m'l}{N}} \\
&= \sum_{m'=0}^{N-1} \left(\hat{f}(m',0) e^{2\pi j \frac{m'l}{N}} + \sum_{n'=1-N}^{-1} \hat{f}(m',n'+N) e^{2\pi j \frac{-n'k+m'l}{N}} \right) \\
&= \sum_{m',n'=0}^{N-1} \hat{f}(m',n') e^{2\pi j \frac{m'l-n'k}{N}} = f(l,-k).
\end{aligned}
$$

13. Let $f \in M_{M\times N}(\mathbb{R})$ be periodically extended, and let $\tilde{f}(k,l) = f(k-k_0, l-l_0)$ for some $k_0, l_0 \in \mathbb{Z}$, please prove $\hat{\tilde{f}} = e^{-2\pi j(\frac{k_0 m}{M} + \frac{l_0 n}{N})} \hat{f}$.

Solution: WLOG assume $k_0 \in \mathbb{Z} \cap [0, M-1]$ and $l_0 \in \mathbb{Z} \cap [0, N-1]$.

- Method 1 (directly): Refer to DFT of a shifted image of **Further properties of DFT** in Section 2.3.
- Method 2 (iDFT):

$$
\begin{aligned}
iDFT(e^{-2\pi j(\frac{k_0 m}{M} + \frac{l_0 n}{N})} \hat{f})(k,l) &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m,n) e^{2\pi j(\frac{m(k-k_0)}{M} + \frac{n(l-l_0)}{N})} \\
&= f(k-k_0, l-l_0).
\end{aligned}
$$

14. Let $f \in M_{M\times N}(\mathbb{R})$ be periodically extended, and let $\tilde{\hat{f}}(m,n) = \hat{f}(m-m_0, n-n_0)$ for some $m_0, n_0 \in \mathbb{Z}$, please prove $\tilde{\hat{f}} = DFT(e^{2\pi j(\frac{km_0}{M} + \frac{ln_0}{N})} f)$.

Solution:

- Method 1 (directly):

$$
\begin{aligned}
\tilde{\hat{f}}(m,n) &= \hat{f}(m-m_0, n-n_0) \\
&= \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k,l) e^{-2\pi j(\frac{k(m-m_0)}{M} + \frac{l(n-n_0)}{N})} \\
&= DFT(e^{2\pi j(\frac{m_0 k}{M} + \frac{n_0 l}{N})} f)(m,n).
\end{aligned}
$$

- Method 2 (iDFT):

$$
\begin{aligned}
iDFT(\tilde{\hat{f}})(k,l) &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \tilde{\hat{f}}(m,n) e^{2\pi j(\frac{mk}{M} + \frac{nl}{N})} \\
&= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m-m_0, n-n_0) e^{2\pi j(\frac{mk}{M} + \frac{nl}{N})} \\
&= \sum_{m'=-m_0}^{M-1-m_0} \sum_{n'=-n_0}^{N-1-n_0} \hat{f}(m',n') e^{2\pi j(\frac{(m'+m_0)k}{M} + \frac{(n'+n_0)l}{N})} \\
&= e^{2\pi j(\frac{m_0 k}{M} + \frac{n_0 l}{N})} f(k,l).
\end{aligned}
$$

15. Please prove that the rank k approximation is the optimal approximation for rank k matrix in sense of Frobenius norm. That is, given a rank r matrix $A \in M_{n\times m}(\mathbb{R})$, for any rank k matrix $B \in M_{n\times m}(\mathbb{R})$, we have

$$\|A - B\|_F \geq \|A - A_k\|_F,$$

where $A_k = \sum_{i=1}^{k} \sigma_i \vec{u_i}\vec{v_i}^T$ is the rank k approximation of $A = \sum_{i=1}^{r} \sigma_i \vec{u_i}\vec{v_i}^T$ and $k = 1, 2, \cdots, r$.

Solution:

This proof is updated to be the proof in Tutorial 6.

Before proving the desired result, we first prove a result:

$$||A - A_k||_2 \leq ||A - B||_2$$

where $||\cdot||_2$ is defined to be

$$\begin{aligned}||C||_2 &:= \sup_{\vec{x}\in\mathbb{R}^N} \frac{||C\vec{x}||_2}{||\vec{x}||_2}\\ &= \sup_{\substack{\vec{x}\in\mathbb{R}^N\\ ||\vec{x}||_2=1}} ||C\vec{x}||_2\\ &= \sigma_1(C),\end{aligned}$$

where $\sigma_i(C)$ is the i-th singular value of C.

Note that since B is of rank k, we can rewrite B as XY^T, where $X \in \mathbb{R}^{M\times k}$ and $Y \in \mathbb{R}^{N\times k}$. (You may consider the SVD of $B = PSQ^T$, take X to be the first k columns of PS, and take Y to be the first k columns of Q.)

Let $\vec{v_1}, \cdots, \vec{v_{k+1}}$ be the first $k+1$ columns of V. Since the span of these vectors has dimension $k+1$, and the Y^T is of rank k, there must be a non trivial linear combination $\vec{w} = \gamma_1\vec{v_1} + \cdots + \gamma_{k+1}\vec{v_{k+1}}$ such that $Y^T\vec{w} = \vec{0}$. Assume further that $||\vec{w}||_2 = 1$.

Then

$$\begin{aligned}||A - B||_2^2 &= \sup_{\substack{\vec{x}\in\mathbb{R}^N\\ ||\vec{x}||_2=1}} ||(A - B)\vec{x}||_2^2\\ &\geq ||(A - B)\vec{w}||_2^2\\ &= ||U\Sigma V^T\vec{w} - XY^T\vec{w}||_2^2\\ &= ||\Sigma(\gamma_1\vec{e_1} + \cdots + \gamma_{k+1}\vec{e_{k+1}})||_2^2\\ &= \gamma_1^2\sigma_1^2 + \cdots + \gamma_{k+1}^2\sigma_{k+1}^2\\ &\geq \sigma_{k+1}^2(\gamma_1^2 + \cdots + \gamma_{k+1}^2)\\ &= ||A - A_k||_2^2\end{aligned}$$

Then back to the proof for the F-norm.

Suppose $A = A' + A''$. Note by the triangle inequality for matrix 2-norm,

$$\sigma_1(M_1 + M_2) \leq \sigma_1(M_1) + \sigma_1(M_2)$$

Then for $i, j \geq 1$,

$$\begin{aligned}\sigma_i(A') + \sigma_j(A'') &= \sigma_1(A' - A'_{i-1}) + \sigma_1(A'' - A''_{j-1})\\ &\geq \sigma_1(A' - A'_{i-1} + A'' - A''_{j-1})\\ &\geq \sigma_1(A - A_{i+j-2})\\ &= \sigma_{i+j-1}(A)\end{aligned}$$

where the second inequality makes use of the fact proved in matrix 2-norm.$A'_{i-1} + A''_{j-1}$ is a matrix of at most rank $i + j - 2$. So the rank $i + j - 2$ approximation A_{i+j-2} of SVD minimize σ_1.

Then take $A' = A - B, A'' = B$. Choose $1 \leq i \leq \min\{n, m\}, j = k + 1$,

$$\sigma_i(A - B) + \sigma_{k+1}(B) = \sigma_i(A - B) \geq \sigma_{i+k}(A)$$

Then

$$\begin{aligned} ||A - B||_F^2 &= \sum_{i=1}^{\min\{n,m\}} \sigma_i(A - B)^2 \\ &\geq \sum_{i=1}^{r-k} \sigma_{i+k}(A)^2 \\ &= \sum_{i=k+1}^{r} \sigma_i(A)^2 \\ &= ||A - A_k||_F^2 \end{aligned}$$

which is desired.