MMAT 5390: Mathematical Image Processing

Assignment 3 solutions

1. Suppose $g = (g(k,l))_{0\le k,l\le N-1}$ is an $N \times N$ image, define $\tilde{g} = (\tilde{g}(k,l))_{-2\le k\le N-3, -4-N\le l\le -5}$ as

$$\tilde{g}(k,l) = g(k+2, -5-l) \text{ for } -2 \le k \le N-3 \text{ and } -4-N \le l \le -5.$$

Prove that

$$DFT(\tilde{g})(m,n) = e^{2\pi j\frac{5n-2m}{N}} DFT(g)(m,-n).$$

Solution: For the LHS, we have

$$\begin{aligned}
&DFT(\tilde{g})(m,n)\\
=&\frac{1}{N^2}\sum_{k=-2}^{N-3}\sum_{l=-4-N}^{-5}\tilde{g}(k,l)e^{-2\pi j\frac{km+ln}{N}}\\
=&\frac{1}{N^2}\sum_{k=-2}^{N-3}\sum_{l=-4-N}^{-5}g(k+2,-5-l)e^{-2\pi j\frac{km+ln}{N}}\\
=&\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}g(k,l)e^{-2\pi j\frac{km-ln+2m-5n}{N}}\\
=&e^{2\pi j\frac{5n-2m}{N}}\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}g(k,l)e^{-2\pi j\frac{km-ln}{N}}
\end{aligned}$$

And the RHS is

$$\begin{aligned}
&e^{2\pi j\frac{5n-2m}{N}}DFT(g)(m,-n)\\
=&e^{2\pi j\frac{5n-2m}{N}}\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}g(k,l)e^{-2\pi j\frac{km-ln}{N}}
\end{aligned}$$

Since LHS = RHS, the original equation is proved.

2. Let $f, g \in \mathbb{R}^{M\times N}$ be $M \times N$ images. Prove that $DFT(f \odot g) = DFT(f) * DFT(g)$, where $f \odot g(k,l) = f(k,l)g(k,l)$.

Solution: We have that

$$\begin{aligned}
DFT(f)(m,n) &= \frac{1}{MN}\sum_{k=1}^{M}\sum_{l=1}^{N}f(k,l)e^{-2\pi j(\frac{km}{M}+\frac{ln}{N})},\\
DFT(g)(m,n) &= \frac{1}{MN}\sum_{k=1}^{M}\sum_{l=1}^{N}g(k,l)e^{-2\pi j(\frac{km}{M}+\frac{ln}{N})},\\
DFT(f\odot g)(m,n) &= \frac{1}{MN}\sum_{k=1}^{M}\sum_{l=1}^{N}f(k,l)g(k,l)e^{-2\pi j(\frac{km}{M}+\frac{ln}{N})}.
\end{aligned}$$

On the other side

$$
\begin{aligned}
&DFT(f) * DFT(g)(m,n) \\
=&\sum_{k=1}^{M}\sum_{l=1}^{N}\hat{f}(k,l)\hat{g}(m-k,n-l) \\
=&\sum_{k=1}^{M}\sum_{l=1}^{N}(\frac{1}{MN}\sum_{x=1}^{M}\sum_{y=1}^{N}f(x,y)e^{-2\pi j(\frac{xk}{M}+\frac{yl}{N})})(\frac{1}{MN}\sum_{x'=1}^{M}\sum_{y'=1}^{N}g(x',y')e^{-2\pi j(\frac{x'(m-k)}{M}+\frac{y'(n-l)}{N})}) \\
=&\frac{1}{M^2N^2}\sum_{k,x,x'=1}^{M}\sum_{l,y,y'=1}^{N}f(x,y)g(x',y')e^{-2\pi j(\frac{xk+x'(m-k)}{M}+\frac{yl+y'(n-l)}{N})} \\
=&\frac{1}{M^2N^2}\sum_{x,x'=1}^{M}\sum_{y,y'=1}^{N}f(x,y)g(x',y')e^{-2\pi j(\frac{x'm}{M}+\frac{y'n}{N})}\sum_{k=1}^{M}e^{-2\pi j\frac{(x-x')k}{M}}\sum_{l=1}^{N}e^{-2\pi j\frac{(y-y')l}{N}} \\
=&\frac{1}{M^2N^2}\sum_{x,x'=1}^{M}\sum_{y,y'=1}^{N}f(x,y)g(x',y')e^{-2\pi j(\frac{x'm}{M}+\frac{y'n}{N})}M\delta(x-x')N\delta(y-y') \\
=&\frac{1}{MN}\sum_{x=1}^{M}\sum_{y=1}^{N}f(x,y)g(x,y)e^{-2\pi j(\frac{xm}{M}+\frac{yn}{N})}.
\end{aligned}
$$

Therefore, $DFT(f \odot g) = DFT(f) * DFT(g)$.