MMAT5390 Mathematical Image Processing
Final Practice Solutions

Solutions prepared by TAs, for your reference only.

1. The Butterworth high-pass filter H with radius D_0 and order n is defined as

$$H(u,v) = \frac{1}{1 + (D_0/D(u,v))^n},$$

where $D(u,v) = u^2 + v^2$. Given an image $I = (I(m,n))_{0\le m,n\le 2N}$ and $N > 100$, apply Butterworth high-pass filter on $DFT(I) = (\hat{I}(u,v))_{0\le u,v\le 2N}$ then get $G(u,v)$. Suppose

$$G(3,4) = \frac{1}{2}\hat{I}(3,4) \text{ and } G(2N-6,8) = \frac{16}{17}\hat{I}(2N-6,8),$$

where $\hat{I}(3,4) \neq 0$ and $\hat{I}(2N-6,8) \neq 0$. Find D_0 and n.

Solution: The given information implies

$$H(3,4) = \frac{1}{2} \text{ and } H(2N-6,8) = \frac{16}{17}.$$

After centralization, we have $H(-6,8) = H(2N-6,8) = \frac{16}{17}$ Hence

$$\begin{cases} \dfrac{25^n}{D_0^n + 25^n} &= \dfrac{1}{2}, \\ \dfrac{100^n}{D_0^n + 100^n} &= \dfrac{16}{17}, \end{cases} \text{ and thus } \begin{cases} 25^n &= D_0^n, \\ 4^n \cdot 25^n &= 16 D_0^n. \end{cases}$$

Then $4^n = 16$. Hence $n = 2$ and then $D_0 = 25$.

2. Consider a 4×4 periodically extended image $I = (I(k,l))_{0\le k,l\le 3}$ given by:

$$I = \begin{pmatrix} a & b & a & b \\ b & a & b & a \\ a & b & a & b \\ b & a & b & a \end{pmatrix},$$

where a and b are distinct positive numbers.

Let $h = (h(k,l))_{0\le k,l\le 3} = \frac{1}{8}\begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, which is periodically extended.

The Gaussian low-pass filter H of variance σ^2 is defined by:

$$H(u,v) = exp\left(-\frac{u^2+v^2}{\sigma^2}\right).$$

Let $I_1(u,v) = H_1(u,v)DFT(I)(u,v)$, where the H_1 is the Gaussian low-pass filter of variance ab. Suppose $I_1(2,2) = e^{-\frac{1}{4}}DFT(I)(2,2)$ and $h * I(2,1) = 6$. Find a and b.

Solution: Since $a \neq b$,

$$DFT(I)(2,2) = \frac{a-b}{2} \neq 0.$$

Hence

$$\begin{cases} exp(-\frac{8}{ab}) = exp(-\frac{1}{4}) & \Longrightarrow ab = 32 \\ \frac{1}{8}(4a+4b) = 6 & \Longrightarrow a+b = 12 \end{cases}$$

Hence $(a,b) = (4,8)$ or $(8,4)$.

3. Compute the degradation functions in the frequency domain that correspond to the following $M \times N$ convolution kernels h, i.e. find $H \in M_{M\times N}(\mathbb{C})$ such that

$$DFT(h * f)(u,v) = H(u,v)DFT(f)(u,v)$$

for any periodically extended $f \in M_{M\times N}(\mathbb{R})$:

(a) Assuming integer k satisfies $k \le \min\{\frac{M}{2}, \frac{N}{2}\}$,

$$h_1(x,y) = \begin{cases} \frac{1}{(2k+1)^2} & \text{if } dist(x, M\mathbb{Z}) \le k \text{ and } dist(y, N\mathbb{Z}) \le k, \\ 0 & \text{otherwise;} \end{cases}$$

(b) Letting $r > 1$,

$$h_2(x,y) = \begin{cases} \frac{r}{r+4} & \text{if } D(x,y) = 0, \\ \frac{1}{r+4} & \text{if } D(x,y) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

(c)

$$h_3(x,y) = \begin{cases} \frac{1}{4} & \text{if } D(x,y) = 0, \\ \frac{1}{8} & \text{if } D(x,y) = 1, \\ \frac{1}{16} & \text{if } D(x,y) = 2, \\ 0 & \text{otherwise;} \end{cases}$$

(d)

$$h_4(x,y) = \begin{cases} -4 & \text{if } D(x,y) = 0, \\ 1 & \text{if } D(x,y) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

(e) Letting $a, b \in \mathbb{Z}$ and $T \in \mathbb{N} \setminus \{0\}$ such that $|a|(T-1) < M$ and $|b|(T-1) < N$,

$$h_5(x,y) = \begin{cases} \frac{1}{T} & \text{if } (x,y) \in \{(at, bt) : t = 0, 1, \cdots, T-1\} \\ 0 & \text{otherwise.} \end{cases}$$

Solution: Recall that for any $f \in M_{M\times N}(\mathbb{R})$, $DFT(h * f) = MN \cdot DFT(h) \odot DFT(f)$; hence $H = MN \cdot DFT(h)$.

(a)

$$\begin{aligned} H_1(u,v) &= \sum_{x=-k}^{k} \sum_{y=-k}^{k} \frac{1}{(2k+1)^2} e^{-2\pi j(\frac{ux}{M} + \frac{vy}{N})} = \frac{1}{(2k+1)^2} \sum_{x=-k}^{k} e^{-2\pi j \frac{ux}{M}} \sum_{y=-k}^{k} e^{-2\pi j \frac{vy}{N}} \\ &= \frac{1}{(2k+1)^2} [1 + 2\sum_{x=1}^{k} \cos\frac{2\pi ux}{M}][1 + 2\sum_{y=1}^{k} \cos\frac{2\pi vy}{N}]. \end{aligned}$$

(b)

$$H_2(u,v) = \frac{r}{r+4} + \frac{1}{r+4}(e^{-2\pi j\frac{u}{M}} + e^{2\pi j\frac{u}{M}} + e^{-2\pi j\frac{v}{N}} + e^{2\pi j\frac{v}{N}}) = \frac{r + 2(\cos\frac{2\pi u}{M} + \cos\frac{2\pi v}{N})}{r+4}$$

(c)

$$\begin{aligned} H_3(u,v) &= \frac{1}{4} + \frac{1}{8}(e^{-2\pi j\frac{u}{M}} + e^{2\pi j\frac{u}{M}} + e^{-2\pi j\frac{v}{N}} + e^{2\pi j\frac{v}{N}}) \\ &\quad + \frac{1}{16}(e^{-2\pi j(\frac{u}{M}+\frac{v}{N})} + e^{-2\pi j(\frac{u}{M}-\frac{v}{N})} + e^{-2\pi j(-\frac{u}{M}+\frac{v}{N})} + e^{-2\pi j(-\frac{u}{M}-\frac{v}{N})}) \\ &= \frac{1}{4} + \frac{1}{4}(\cos\frac{2\pi u}{M} + \cos\frac{2\pi v}{N}) + \frac{1}{4}\cos\frac{2\pi u}{M}\cos\frac{2\pi v}{N} \\ &= \frac{1}{4}(\cos\frac{2\pi u}{M} + 1)(\cos\frac{2\pi v}{N} + 1) \\ &= \cos^2\frac{\pi u}{M}\cos^2\frac{\pi v}{N}. \end{aligned}$$

(d)

$$\begin{aligned} H_4(u,v) &= -4 + e^{-2\pi j\frac{u}{M}} + e^{2\pi j\frac{u}{M}} + e^{-2\pi j\frac{v}{N}} + e^{2\pi j\frac{v}{N}} \\ &= -4 + 2\cos\frac{2\pi u}{M} + 2\cos\frac{2\pi v}{N} \\ &= -4(\sin^2\frac{\pi u}{M} + \sin^2\frac{\pi v}{N}). \end{aligned}$$

(e)

$$
\begin{aligned}
H_5(u,v) &= \frac{1}{T}\sum_{t=0}^{T-1} e^{-2\pi j\left(\frac{atu}{M}+\frac{btv}{N}\right)} \\
&= \begin{cases} \frac{1}{T}\cdot\frac{1-e^{-2\pi jT\left(\frac{au}{M}+\frac{bv}{N}\right)}}{1-e^{-2\pi j\left(\frac{au}{M}+\frac{bv}{N}\right)}} & \text{if } \frac{au}{M}+\frac{bv}{N}\notin\mathbb{Z}, \\ 1 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{1}{T}e^{-\pi j(T-1)\left(\frac{au}{M}+\frac{bv}{N}\right)}\frac{e^{\pi jT\left(\frac{au}{M}+\frac{bv}{N}\right)}-e^{-\pi jT\left(\frac{au}{M}+\frac{bv}{N}\right)}}{e^{\pi j\left(\frac{au}{M}+\frac{bv}{N}\right)}-e^{-\pi j\left(\frac{au}{M}+\frac{bv}{N}\right)}} & \text{if } \frac{au}{M}+\frac{bv}{N}\notin\mathbb{Z}, \\ 1 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{1}{T}e^{-\pi j(T-1)\left(\frac{au}{M}+\frac{bv}{N}\right)}\frac{\sin\left(\pi T\left(\frac{au}{M}+\frac{bv}{N}\right)\right)}{\sin\left(\pi\left(\frac{au}{M}+\frac{bv}{N}\right)\right)} & \text{if } \frac{au}{M}+\frac{bv}{N}\notin\mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}
$$

4. For any periodically extended $N\times N$ image f, define

$$
\begin{aligned}
G_x(f)(x,y) &= \frac{1}{4}f(x+1,y)+\frac{1}{2}f(x,y)+\frac{1}{4}f(x-1,y) \\
\text{and } G_y(f)(x,y) &= \frac{1}{4}f(x,y+1)+\frac{1}{2}f(x,y)+\frac{1}{4}f(x,y-1).
\end{aligned}
$$

(a) Find an $N\times N$ image h such that for any periodically extended $N\times N$ image f,

$$h*f = G_x\circ G_y(f).$$

(b) Let $H(u,v)$ be the LPF such that

$$DFT(h*f)(u,v) = H(u,v)DFT(f)(u,v),$$

where h is the convolution kernel from (a). Using H, perform unsharp masking (i.e. $k=1$) on the following periodically extended 4×4 image

$$
f = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
$$

Solution:

(a) Note that

$$G_x(f)(x,y) = h_x * f(x,y) \text{ and } G_y(f)(x,y) = h_y * f(x,y),$$

where

$$
h_x(x,y) = \begin{cases} \frac{1}{2} & \text{if } (x,y)=(0,0) \\ \frac{1}{4} & \text{if } (x,y)=(-1,0) \text{ or } (1,0) \\ 0 & \text{otherwise} \end{cases}
$$

and

$$
h_y(x,y) = \begin{cases} \frac{1}{2} & \text{if } (x,y)=(0,0) \\ \frac{1}{4} & \text{if } (x,y)=(0,-1) \text{ or } (0,1) \\ 0 & \text{otherwise.} \end{cases}
$$

Hence $G_x(G_y(f)) = h_x*(h_y*f) = (h_x*h_y)*f = h*f$, where

$$
\begin{aligned}
h(x,y) &= h_x * h_y(x,y) \\
&= \begin{cases} \frac{1}{4} & \text{if } (x,y)=(0,0) \\ \frac{1}{8} & \text{if } (x,y)=(0,-1) \text{ or } (-1,0) \text{ or } (1,0) \text{ or } (0,1) \\ \frac{1}{16} & \text{if } (x,y)=(-1,-1) \text{ or } (1,-1) \text{ or } (-1,1) \text{ or } (1,1) \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

(b) Recall that $DFT(h*f)(u,v) = N^2 DFT(h)(u,v)DFT(f)(u,v)$.

Hence to perform unsharp masking on $f \in M_{4\times 4}$,

$$\begin{aligned}
H &= 16DFT(h)\\
&= \frac{1}{16}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}\begin{pmatrix}4 & 2 & 0 & 2\\ 2 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\\ 2 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}\\
&= \frac{1}{16}\begin{pmatrix}8 & 4 & 0 & 4\\ 4 & 2 & 0 & 2\\ 0 & 0 & 0 & 0\\ 4 & 2 & 0 & 2\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}\\
&= \frac{1}{16}\begin{pmatrix}16 & 8 & 0 & 8\\ 8 & 4 & 0 & 4\\ 0 & 0 & 0 & 0\\ 8 & 4 & 0 & 4\end{pmatrix} = \frac{1}{4}\begin{pmatrix}4 & 2 & 0 & 2\\ 2 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\\ 2 & 1 & 0 & 1\end{pmatrix}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
DFT(f) &= \frac{1}{16}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}\begin{pmatrix}4 & 1 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}\\
&= \frac{1}{16}\begin{pmatrix}6 & 1 & 0 & 1\\ 4 & 1 & 0 & 1\\ 2 & 1 & 0 & 1\\ 4 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}\\
&= \frac{1}{16}\begin{pmatrix}8 & 6 & 4 & 6\\ 6 & 4 & 2 & 4\\ 4 & 2 & 0 & 2\\ 6 & 4 & 2 & 4\end{pmatrix} = \frac{1}{8}\begin{pmatrix}4 & 3 & 2 & 3\\ 3 & 2 & 1 & 2\\ 2 & 1 & 0 & 1\\ 3 & 2 & 1 & 2\end{pmatrix}.
\end{aligned}$$

$\tilde{F}(u,v) = DFT(f)(u,v)[2 - H(u,v)]$ and thus

$$\begin{aligned}
\tilde{F} &= \frac{1}{32}\begin{pmatrix}4 & 3 & 2 & 3\\ 3 & 2 & 1 & 2\\ 2 & 1 & 0 & 1\\ 3 & 2 & 1 & 2\end{pmatrix} \odot \begin{pmatrix}4 & 6 & 8 & 6\\ 6 & 7 & 8 & 7\\ 8 & 8 & 8 & 8\\ 6 & 7 & 8 & 7\end{pmatrix}\\
&= \frac{1}{32}\begin{pmatrix}16 & 18 & 16 & 18\\ 18 & 14 & 8 & 14\\ 16 & 8 & 0 & 8\\ 18 & 14 & 8 & 14\end{pmatrix} = \frac{1}{16}\begin{pmatrix}8 & 9 & 8 & 7\\ 9 & 7 & 4 & 7\\ 8 & 4 & 0 & 4\\ 9 & 7 & 4 & 7\end{pmatrix}\\
\text{and } \tilde{f} &= \frac{1}{16}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{pmatrix}\begin{pmatrix}8 & 9 & 8 & 7\\ 9 & 7 & 4 & 7\\ 8 & 4 & 0 & 4\\ 9 & 7 & 4 & 7\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{pmatrix}\\
&= \frac{1}{16}\begin{pmatrix}34 & 27 & 16 & 27\\ 0 & 5 & 8 & 5\\ -2 & -1 & 0 & -1\\ 0 & 5 & 8 & 5\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{pmatrix}\\
&= \frac{1}{16}\begin{pmatrix}104 & 18 & -4 & 18\\ 18 & -8 & -2 & -8\\ -4 & -2 & 0 & -2\\ 18 & -8 & -2 & -8\end{pmatrix}\\
&= \frac{1}{8}\begin{pmatrix}52 & 9 & -2 & 9\\ 9 & -4 & -1 & -4\\ -2 & -1 & 0 & -1\\ 9 & -4 & -1 & -4\end{pmatrix}.
\end{aligned}$$

5. Let $W_N(n,k) = \dfrac{1}{\sqrt{N}} e^{2\pi j \frac{nk}{N}}$ for $0 \le n, k \le N-1$ and $W = W_N \otimes W_N$. Suppose $N = 4$, we have following problems:

(a) Prove that $W^{-1} = \overline{W_N} \otimes \overline{W_N}$.

(b) Show that $W^{-1}\mathcal{S}(f) = N\mathcal{S}(\hat{f})$ for any $f \in M_{N\times N}(\mathbb{C})$, where $\hat{f} = DFT(f)$.

Solution: Method 1 (for general N):

(a) Recall that the definition of Kronecker product is

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1N}B \\ a_{21}B & a_{22}B & \cdots & a_{2N}B \\ \vdots & \vdots & & \vdots \\ a_{N1}B & a_{N2}B & \cdots & a_{NN}B \end{pmatrix}$$

We have

$$W = W_N \otimes W_N = \begin{pmatrix} \frac{1}{\sqrt{N}}e^{2\pi j\frac{0\cdot 0}{N}}\cdot W_N & \frac{1}{\sqrt{N}}e^{2\pi j\frac{0\cdot 1}{N}}\cdot W_N & \cdots & \frac{1}{\sqrt{N}}e^{2\pi j\frac{0\cdot (N-1)}{N}}\cdot W_N \\ \frac{1}{\sqrt{N}}e^{2\pi j\frac{1\cdot 0}{N}}\cdot W_N & \frac{1}{\sqrt{N}}e^{2\pi j\frac{1\cdot 1}{N}}\cdot W_N & \cdots & \frac{1}{\sqrt{N}}e^{2\pi j\frac{1\cdot (N-1)}{N}}\cdot W_N \\ \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{N}}e^{2\pi j\frac{(N-1)\cdot 0}{N}}\cdot W_N & \frac{1}{\sqrt{N}}e^{2\pi j\frac{(N-1)\cdot 1}{N}}\cdot W_N & \cdots & \frac{1}{\sqrt{N}}e^{2\pi j\frac{(N-1)\cdot (N-1)}{N}}\cdot W_N \end{pmatrix}$$

And rewrite it as $W = \left(\frac{1}{\sqrt{N}}e^{2\pi j\frac{n\cdot k}{N}}\cdot W_N\right)_{0\le n,k\le N-1}$

Since $\overline{W_N} = \left(\dfrac{1}{\sqrt{N}}e^{-2\pi j\frac{nk}{N}}\right)_{0\le n,k\le N-1}$, we know that

$$\overline{W_N}\otimes\overline{W_N} = \begin{pmatrix} \frac{1}{\sqrt{N}}e^{-2\pi j\frac{0\cdot 0}{N}}\cdot \overline{W_N} & \frac{1}{\sqrt{N}}e^{-2\pi j\frac{0\cdot 1}{N}}\cdot \overline{W_N} & \cdots & \frac{1}{\sqrt{N}}e^{-2\pi j\frac{0\cdot (N-1)}{N}}\cdot \overline{W_N} \\ \frac{1}{\sqrt{N}}e^{-2\pi j\frac{1\cdot 0}{N}}\cdot \overline{W_N} & \frac{1}{\sqrt{N}}e^{-2\pi j\frac{1\cdot 1}{N}}\cdot \overline{W_N} & \cdots & \frac{1}{\sqrt{N}}e^{-2\pi j\frac{1\cdot (N-1)}{N}}\cdot \overline{W_N} \\ \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{N}}e^{-2\pi j\frac{(N-1)\cdot 0}{N}}\cdot \overline{W_N} & \frac{1}{\sqrt{N}}e^{-2\pi j\frac{(N-1)\cdot 1}{N}}\cdot \overline{W_N} & \cdots & \frac{1}{\sqrt{N}}e^{-2\pi j\frac{(N-1)\cdot (N-1)}{N}}\cdot \overline{W_N} \end{pmatrix}$$

And rewrite it as $\overline{W_N}\otimes\overline{W_N} = \left(\frac{1}{\sqrt{N}}e^{-2\pi j\frac{n\cdot k}{N}}\cdot \overline{W_N}\right)_{0\le n,k\le N-1}$

Using block-matrix multiplication, we can calculate

$$\begin{aligned} W\cdot(\overline{W_N}\otimes\overline{W_N}) &= \left(\frac{1}{\sqrt{N}}e^{2\pi j\frac{n\cdot k}{N}}\cdot W_N\right)_{0\le n,k\le N-1}\left(\frac{1}{\sqrt{N}}e^{-2\pi j\frac{n\cdot k}{N}}\cdot \overline{W_N}\right)_{0\le n,k\le N-1} \\ &= \left(\sum_{p=0}^{N-1}\frac{1}{\sqrt{N}}e^{2\pi j\frac{n\cdot p}{N}}\cdot W_N\cdot\frac{1}{\sqrt{N}}e^{-2\pi j\frac{p\cdot k}{N}}\cdot\overline{W_N}\right)_{0\le n,k\le N-1} \\ &= \left(\sum_{p=0}^{N-1}\frac{1}{N}e^{2\pi j\frac{(n-k)\cdot p}{N}}\cdot W_N\overline{W_N}\right)_{0\le n,k\le N-1} \\ &= I_{N^2} \end{aligned}$$

Therefore, $W^{-1} = \overline{W_N}\otimes\overline{W_N}$.

(b) Note that $f = (f_{i,j})_{0\le i,j\le N-1} \in M_{N\times N}(\mathbb{C})$ then

$$\mathcal{S}(f) = \begin{pmatrix} f_{0,0} & f_{1,0} & \cdots & f_{N-1,0} & \cdots & f_{0,N-1} & f_{1,N-1} & \cdots & f_{N-1,N-1}\end{pmatrix}^T \in M_{N^2\times 1}(\mathbb{C})$$

From (a) we know that $W^{-1} = \overline{W_N}\otimes\overline{W_N} = \left(\frac{1}{\sqrt{N}}e^{-2\pi j\frac{n\cdot k}{N}}\cdot\overline{W_N}\right)_{0\le n,k\le N-1} \in M_{N^2\times N^2}(\mathbb{C})$

Then, $W^{-1}\mathcal{S}(f) \in M_{N^2\times 1}(\mathbb{C})$, and its l-th entry is

$$
\begin{aligned}
\left(W^{-1}\mathcal{S}(f)\right)_l &= W^{-1}(l,:)\cdot\mathcal{S}(f)\\
&= \begin{pmatrix}
\frac{1}{\sqrt{N}}e^{-2\pi j\frac{\lfloor\frac{l}{N}\rfloor\cdot 0}{N}}\cdot\frac{1}{\sqrt{N}}e^{-2\pi j\frac{\mathrm{mod}_N(l)\cdot 0}{N}}\\
\frac{1}{\sqrt{N}}e^{-2\pi j\frac{\lfloor\frac{l}{N}\rfloor\cdot 0}{N}}\cdot\frac{1}{\sqrt{N}}e^{-2\pi j\frac{\mathrm{mod}_N(l)\cdot 1}{N}}\\
\vdots\\
\frac{1}{\sqrt{N}}e^{-2\pi j\frac{\lfloor\frac{l}{N}\rfloor\cdot 0}{N}}\cdot\frac{1}{\sqrt{N}}e^{-2\pi j\frac{\mathrm{mod}_N(l)\cdot (N-1)}{N}}\\
\vdots\\
\frac{1}{\sqrt{N}}e^{-2\pi j\frac{\lfloor\frac{l}{N}\rfloor\cdot (N-1)}{N}}\cdot\frac{1}{\sqrt{N}}e^{-2\pi j\frac{\mathrm{mod}_N(l)\cdot 0}{N}}\\
\frac{1}{\sqrt{N}}e^{-2\pi j\frac{\lfloor\frac{l}{N}\rfloor\cdot (N-1)}{N}}\cdot\frac{1}{\sqrt{N}}e^{-2\pi j\frac{\mathrm{mod}_N(l)\cdot 1}{N}}\\
\vdots\\
\frac{1}{\sqrt{N}}e^{-2\pi j\frac{\lfloor\frac{l}{N}\rfloor\cdot (N-1)}{N}}\cdot\frac{1}{\sqrt{N}}e^{-2\pi j\frac{\mathrm{mod}_N(l)\cdot (N-1)}{N}}
\end{pmatrix}^T
\begin{pmatrix}
f_{0,0}\\ f_{1,0}\\ \vdots\\ f_{N-1,0}\\ \vdots\\ f_{0,N-1}\\ f_{1,N-1}\\ \vdots\\ f_{N-1,N-1}
\end{pmatrix}\\
&= \sum_{p=0}^{N-1}\sum_{q=0}^{N-1}\frac{1}{N}e^{-2\pi j\frac{\lfloor\frac{l}{N}\rfloor\cdot q+\mathrm{mod}_N(l)\cdot p}{N}}f_{p,q}
\end{aligned}
$$

From $\hat{f}_{p,q} = \frac{1}{N^2}\sum_{\alpha,\beta=0}^{N-1} f_{\alpha,\beta}\cdot e^{-2\pi j\frac{p\alpha+q\beta}{N}}$, we have $\mathcal{S}(\hat{f}) = \begin{pmatrix}\hat{f}_{0,0}\\ \hat{f}_{1,0}\\ \vdots\\ \hat{f}_{N-1,0}\\ \vdots\\ \hat{f}_{0,N-1}\\ \hat{f}_{1,N-1}\\ \vdots\\ \hat{f}_{N-1,N-1}\end{pmatrix} \in M_{N^2\times 1}(\mathbb{C})$

and $(N\cdot(\mathcal{S}(f)))_l = N\hat{f}(\mathrm{mod}_N(l), \lfloor\frac{l}{N}\rfloor) = N\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}\frac{1}{N^2}e^{-2\pi j\frac{\lfloor\frac{l}{N}\rfloor\cdot q+\mathrm{mod}_N(l)\cdot p}{N}}f_{p,q}$

Therefore, $W^{-1}\mathcal{S}(f) = N\mathcal{S}(\hat{f})$ for any $f\in M_{N\times N}(\mathbb{C})$, where $\hat{f} = DFT(f)$.

Method 2 (directly compute when $N=4$):

(a) We can directly calculate that

$$W_4 = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{pmatrix}$$

and

$$\overline{W_4} = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}.$$

It can be checked that $W_4\overline{W_4} = I_4$, which means $\overline{W_4} = W_4^{-1}$. Therefore

$$W^{-1} = W_4^{-1}\otimes W_4^{-1} = \overline{W_4}\otimes\overline{W_4}.$$

(b) By the matrix form of DFT, we have that

$$\hat{f} = DFT(f) = \frac{1}{16}(2\overline{W_4})f(2\overline{W_4}) = \frac{1}{4}\overline{W_4}f\overline{W_4}.$$

So it can be treated as a spearable linear transformation with transformation matrix

$$\overline{W_4}^T\otimes(\frac{1}{4}\overline{W_4}) = \frac{1}{4}W^{-1}.$$

Then

$$W^{-1}\mathcal{S}(f) = 4\left(\frac{1}{4}W^{-1}\mathcal{S}(f)\right) = 4\mathcal{S}(\hat{f}).$$

6. Let $f = (f_{ij})_{0\le i,j\le N-1} \in M_{N\times N}(\mathbb{R})$ be a clean image. Suppose f is blurred to g under a motion which is given by:
$$g(x,y) = \sum_{t=0}^{3} f(x+t,y).$$
Show that $DFT(g)(u,v) = H(u,v)DFT(f)(u,v)$ and find $H(u,v)$.

Solution:
$$\begin{aligned}
DFT(g)(u,v) &= \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} g(x,y)e^{-2\pi j\frac{ux+vy}{N}}\\
&= \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sum_{t=0}^{3} f(x+t,y)e^{-2\pi j\frac{ux+vy}{N}}\\
&= \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sum_{t=0}^{3} f(x+t,y)e^{-2\pi j\frac{u(x+t)+vy}{N}}e^{-2\pi j\frac{-ut}{N}}\\
&= \frac{1}{N^2}\sum_{t=0}^{3}\sum_{x'=t}^{N-1+t}\sum_{y=0}^{N-1} f(x',y)e^{-2\pi j\frac{ux'+vy}{N}}e^{-2\pi j\frac{-ut}{N}}\\
&= \left(\sum_{t=0}^{3} e^{2\pi j\frac{ut}{N}}\right)\left[\frac{1}{N^2}\sum_{x'=0}^{N-1}\sum_{y=0}^{N-1} f(x',y)e^{-2\pi j\frac{ux'+vy}{N}}\right]\\
&= \begin{cases} \dfrac{1-e^{8\pi j\frac{u}{N}}}{1-e^{2\pi j\frac{u}{N}}}DFT(f)(u,v) & \text{if } u\neq 0,\\ 4\,DFT(f)(u,v) & \text{otherwise,}\end{cases}
\end{aligned}$$

Hence $H(u,v) = \begin{cases} \dfrac{1-e^{8\pi j\frac{u}{N}}}{1-e^{2\pi j\frac{u}{N}}} & \text{if } u\neq 0,\\ 4 & \text{otherwise,}\end{cases}$

7. Given $N^2\times N^2$ block-circulant real matrices D and L, $N\times N$ image g and fixed parameter $\varepsilon > 0$, the constrained least square filtering aims to find $f\in M_{N\times N}$ that minimizes:
$$E(f) = [L\mathcal{S}(f)]^T[L\mathcal{S}(f)]$$
subject to the constraint:
$$[\mathcal{S}(g) - D\mathcal{S}(f)]^T[\mathcal{S}(g) - D\mathcal{S}(f)] = \varepsilon,$$
where $\mathcal{S}$ is the stacking operator. Let $W = W_N\otimes W_N$, where $W_N(n,k) = \frac{1}{\sqrt{N}}e^{2\pi j\frac{nk}{N}}$, we have D and L is diagonalizable by W, i.e. $\Lambda_D = W^{-1}DW$ and $\Lambda_L = W^{-1}LW$ is diagonal. Given that the optimal solution f that solves the constrained least square problem satisfies
$$[\lambda D^T D + L^T L]\mathcal{S}(f) = \lambda D^T\mathcal{S}(g)$$
for some parameter λ. Find $DFT(f)$ in terms of $DFT(g)$, $DFT(h)$, $DFT(p)$ and λ, where $L\mathcal{S}(f) = \mathcal{S}(p*f)$ and $D\mathcal{S}(f) = \mathcal{S}(h*f)$ for any $f\in M_{N\times N}(\mathbb{R})$.

Solution:

Let $\vec{f} = \mathcal{S}(f)$ and $\vec{g} = \mathcal{S}(g)$. It's easy to know $D = W\Lambda_D W^{-1}$ and $L = W\Lambda_L W^{-1}$. Hence
$$\begin{aligned}
(\lambda D^T D + L^T L)\vec{f} &= (\lambda D^* D + L^* L)\vec{f}\\
&= \{\lambda[W\Lambda_D W^{-1}]^* W\Lambda_D W^{-1} + [W\Lambda_L W^{-1}]^* W\Lambda_L W^{-1}\}\vec{f}\\
&= [\lambda W\Lambda_D^*\Lambda_D W^{-1} + W\Lambda_L^*\Lambda_L W^{-1}]\vec{f}\\
&= W(\lambda\Lambda_D^*\Lambda_D + \Lambda_L^*\Lambda_L)W^{-1}\vec{f}\\
&= W(\lambda\Lambda_D^*\Lambda_D + \Lambda_L^*\Lambda_L)W^{-1}N\mathcal{S}(DFT(f));
\end{aligned}$$
on the other hand,
$$\begin{aligned}
\lambda D^T\vec{g} &= \lambda D^*\vec{g}\\
&= \lambda W\Lambda_D^* W^{-1}\vec{g}\\
&= \lambda W\Lambda_D^* N\,\mathcal{S}(DFT(g)).
\end{aligned}$$

Hence $(\lambda \Lambda_D^* \Lambda_D + \Lambda_L^* \Lambda_L)\mathcal{S}(DFT(f)) = \lambda \Lambda_D^* \mathcal{S}(DFT(g))$.

From Theorem 1 in lecture 16, we know that for any $k, l, x, y \in \{0, 1, \cdots, N-1\}$

$$\Lambda_D(x+kN, y+lN) = \begin{cases} N^2\, DFT(h)(x,k) & \text{if } k=l \text{ and } x=y, \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda_L(x+kN, y+lN) = \begin{cases} N^2\, DFT(p)(x,k) & \text{if } k=l \text{ and } x=y, \\ 0 & \text{otherwise} \end{cases}$$

By comparing each pair of entries, we have

$$(\lambda N^4 |DFT(h)(u,v)|^2 + N^4 |DFT(p)(u,v)|^2) DFT(f)(u,v) = \lambda N^2 DFT(h)(u,v) DFT(g)(u,v),$$

which yields

$$DFT(f)(u,v) = \frac{\lambda DFT(h)(u,v) DFT(g)(u,v)}{N^2(\lambda |DFT(h)(u,v)|^2 + |DFT(p)(u,v)|^2)}.$$

8. Given $g \in M_{N\times N}(\mathbb{R})$, block circulant $D, L_1, L_2 \in M_{N^2\times N^2}(\mathbb{R})$ and $\varepsilon > 0$, we aim to minimize $\|L_1\vec{f}\|_2^2 + \|L_2\vec{f}\|_2^2$ subject to $\|\vec{g} - D\vec{f}\|_2^2 = \varepsilon$ over $f \in M_{N\times N}(\mathbb{R})$, where $\vec{f} = \mathcal{S}(f)$ and $\vec{g} = \mathcal{S}(g)$ vectorized by the stack operator $\mathcal{S}$.

   Given Lagrange multiplier λ for the equality constraint, show that if f is a minimizer of the above constrained minimization problem, then

   $$(\lambda D^T D + L_1^T L_1 + L_2^T L_2)\vec{f} = \lambda D^T \vec{g}.$$

   Please prove your answer with details.

   Solution:

   Please check that $\dfrac{\partial \vec{f}^T \vec{a}}{\partial \vec{f}} = \vec{a}$, $\dfrac{\partial \vec{b}^T \vec{f}}{\partial \vec{f}} = \vec{b}$ and $\dfrac{\partial \vec{f}^T A \vec{f}}{\partial \vec{f}} = (A + A^T)\vec{f}$.

   We know the minimizer must satisfy

   $$\mathcal{D} = \frac{\partial}{\partial \vec{f}}[\vec{f}^T L_1^T L_1 \vec{f} + \vec{f}^T L_2^T L_2 \vec{f} + \lambda(\vec{g} - D\vec{f})^T(\vec{g} - D\vec{f})] = 0$$

   where λ is the Lagrange multiplier. Therefore,

   $$\begin{aligned} &\mathcal{D} = 0 \\ &\Rightarrow 2(L_1^T L_1 + L_2^T L_2)\vec{f} + \lambda(-D^T\vec{g} - D^T\vec{g} + 2D^T D\vec{f}) = 0 \\ &\Rightarrow (\lambda D^T D + L_1^T L_1 + L_2^T L_2)\vec{f} = \lambda D^T \vec{g}. \end{aligned}$$

9. Given a 2D simple connected domain D and a noisy image $I : D \to \mathbb{R}$. Assume $I = 0$ on the boundary of D. We consider the following image denoising model to restore the original clean image $f : D \to \mathbb{R}$ that minimizes:

   $$E(f) = \int_D (f(x,y) - I(x,y))^2 \, dxdy + \int_D \sqrt{|\nabla f(x,y)|^2 + \epsilon} \, dxdy$$

   where small parameter $\epsilon > 0$.

   (a) If f minimizes $E(f)$, show that f should satisfy the following conditions:

   $$2f(x,y) - 2I(x,y) - \nabla \cdot \left(\frac{\nabla f(x,y)}{\sqrt{|\nabla f(x,y)|^2 + \epsilon}} \right) = 0 \text{ for } (x,y) \in D.$$

   (b) Derive an iterative scheme, which updates f_n to f_{n+1} with time step $\tau > 0$, to minimize $E(f)$.

   Solution:

(a) Suppose f minimizes $E(f)$, then for any $v : D \to \mathbb{R}$,

$$\begin{aligned}
0 &= \frac{d}{dt}E(f+tv)\bigg|_{t=0} \\
&= \int_D \frac{d}{dt}\bigg|_{t=0}\left[(f+tv-I)^2 + \sqrt{|\nabla(f+tv)|^2+\epsilon}\right] dxdy \\
&= \int_D \left[2(f-I)v + 2tv^2 + \frac{(\nabla f + t\nabla v)\cdot\nabla v}{\sqrt{|\nabla f + t\nabla v|^2+\epsilon}}\right]\bigg|_{t=0} dxdy \\
&= \int_D \left[2(f-I)v + \frac{\nabla f\cdot\nabla v}{\sqrt{|\nabla f|^2+\epsilon}}\right] dxdy \\
&= \int_D \left[2(f-I) - \nabla\cdot\left(\frac{\nabla f}{\sqrt{|\nabla f|^2+\epsilon}}\right)\right] vdxdy + \int_{\partial D}\langle\frac{\nabla f}{\sqrt{|\nabla f|^2+\epsilon}}, \vec{n}\rangle vds
\end{aligned}$$

We have $I = 0$ on ∂D, so $\vec{n} = 0$ holds on ∂D and the last term can be removed as

$$0 = \int_D \left[2(f-I) - \nabla\cdot\left(\frac{\nabla f}{\sqrt{|\nabla f|^2+\epsilon}}\right)\right] vdxdy.$$

Since the above equation holds for any v, it could be

$$2f(x,y) - 2I(x,y) - \nabla\cdot\left(\frac{\nabla f(x,y)}{\sqrt{|\nabla f(x,y)|^2+\epsilon}}\right) = 0 \text{ for } (x,y)\in D.$$

(b) Based on the derivation above, dropping the assumption that f is a minimizer of E, we have:

$$\begin{aligned}
\frac{d}{dt}E(f+tv) &= \int_D \frac{d}{dt}\left[(f+tv-I)^2 + \sqrt{|\nabla(f+tv)|^2+\epsilon}\right] dxdy \\
&= \int_D 2(f-I)v + 2tv^2 + \frac{(\nabla f + t\nabla v)\cdot\nabla v}{\sqrt{|\nabla f + t\nabla v|^2+\epsilon}} dxdy \\
&\approx \int_D 2(f-I)v + \frac{\nabla f\cdot\nabla v}{\sqrt{|\nabla f|^2+\epsilon}} dxdy \quad \text{for small } t \\
&= \int_D \left[2(f-I) - \nabla\cdot\left(\frac{\nabla f}{\sqrt{|\nabla f|^2+\epsilon}}\right)\right] vdxdy
\end{aligned}$$

For v to be a descent direction, we need $\frac{d}{dt}E(f+tv) < 0$.
So choosing $v = -\left[2(f-I) - \nabla\cdot\left(\frac{\nabla f}{\sqrt{|\nabla f|^2+\epsilon}}\right)\right]$ on D, the integrals will be both less than 0. Hence a descent direction is:

$$-2f(x,y) + 2I(x,y) + \nabla\cdot\left(\frac{\nabla f(x,y)}{\sqrt{|\nabla f(x,y)|^2+\epsilon}}\right) \quad \text{for } (x,y)\in D,$$

and thus $E(f)$ can be iteratively minimized by updating f:

$$f^{n+1}(x,y) = f^n(x,y) - \tau\left[2f(x,y) - 2I(x,y) - \nabla\cdot\left(\frac{\nabla f(x,y)}{\sqrt{|\nabla f(x,y)|^2+\epsilon}}\right)\right] \text{ if } (x,y)\in D$$

for a small time step $\tau > 0$.

10. Given a noisy image $I : D \to \mathbb{R}$, we consider the following image denoising model to restore the original clean image $f : D \to \mathbb{R}$ that minimizes:

$$E(f) = \int_D (f(x,y) - I(x,y))^2 \, dxdy + \int_D |\nabla f(x,y)|^2 \, dxdy.$$

where $|\nabla f(x,y)|^2 = (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$. Assume $I = 0$ on the boundary of D.

(a) Derive an iterative scheme, which updates f^n to f^{n+1} with time step $\tau > 0$, to minimize $E(f)$. This is the same model as discussed in the lectures. Please show all your steps with details, including detailed explanations on why E is iteratively decreasing. Missing detailed steps will result in mark deductions.

(b) If E is modified to $\widetilde{E}$ defined as follows:

$$\widetilde{E}(f) = \int_D \sqrt{(f(x,y) - I(x,y))^2 + \epsilon^2}\; dxdy + \int_D \sqrt{|\nabla f(x,y)|^2 + \epsilon^2}\; dxdy,$$

where $\epsilon > 0$ is a small parameter bigger than 0. Derive an iterative scheme, which updates f^n to f^{n+1} with time step $\tau > 0$, to minimize $\widetilde{E}(f)$. Please show all your steps with details, including detailed explanations on why $\widetilde{E}$ is iteratively decreasing. Missing detailed steps will result in mark deductions.

Solution:

(a) Suppose f minimizes $E(f)$, then for any $v : D \to \mathbb{R}$,

$$\begin{aligned}
0 &= \frac{d}{dt} E(f+tv)\bigg|_{t=0} \\
&= \int_D \frac{d}{dt}\bigg|_{t=0} \left[(f+tv-I)^2 + \langle \nabla(f+tv), \nabla(f+tv)\rangle\right] dxdy \\
&= \int_D 2\left[(f-I)v + tv^2 + \langle \nabla f, \nabla v\rangle + t\langle \nabla v, \nabla v\rangle\right]\bigg|_{t=0} dxdy \\
&= \int_D 2\left[(f-I)v + \nabla f \cdot \nabla v\right] dxdy \\
&= \int_D 2\left[(f-I) - \nabla\cdot\nabla f\right] v dxdy + \int_{\partial D} 2\langle \nabla f, \vec{n}\rangle v ds
\end{aligned}$$

We have $I = 0$ on ∂D, so $\vec{n} = 0$ there, the equation becomes

$$\int_D 2\left[(f-I) - \Delta f\right] v dxdy = 0.$$

Since the above equation holds for any v, it must be

$$f(x,y) - I(x,y) - \Delta f(x,y) = 0 \text{ for } (x,y) \in D$$

Refer to the argument in 9(b), a descent direction is:

$$-2f(x,y) + 2I(x,y) + 2\Delta f(x,y) \text{ for } (x,y) \in D,$$

and thus $E(f)$ can be iteratively minimized by updating f:

$$f^{n+1}(x,y) = f^n(x,y) - 2\tau\left[f(x,y) - I(x,y) - \Delta f(x,y)\right] \text{if } (x,y) \in D$$

for a small time step $\tau > 0$.

(b) Suppose f minimizes $\tilde{E}(f)$, then for any $v : D \to \mathbb{R}$,

$$\begin{aligned}
0 &= \frac{d}{dt} \tilde{E}(f+tv)\bigg|_{t=0} \\
&= \int_D \frac{d}{dt}\bigg|_{t=0} \left[\sqrt{(f+tv-I)^2 + \epsilon^2} + \sqrt{|\nabla(f+tv)|^2 + \epsilon^2}\right] dxdy \\
&= \int_D \left[\frac{(f-I)v + tv^2}{\sqrt{(f+tv-I)^2+\epsilon^2}} + \frac{(\nabla f + t\nabla v)\cdot\nabla v}{\sqrt{|\nabla f + t\nabla v|^2 + \epsilon^2}}\right]\bigg|_{t=0} dxdy \\
&= \int_D \left[\frac{(f-I)v}{\sqrt{(f-I)^2+\epsilon^2}} + \frac{\nabla f\cdot\nabla v}{\sqrt{|\nabla f|^2+\epsilon}}\right] dxdy \\
&= \int_D \left[\frac{f-I}{\sqrt{(f-I)^2+\epsilon^2}} - \nabla\cdot\left(\frac{\nabla f}{\sqrt{|\nabla f|^2+\epsilon}}\right)\right] vdxdy
\end{aligned}$$

Since the above equation holds for any v, it must be

$$\frac{f-I}{\sqrt{(f-I)^2+\epsilon^2}} - \nabla\cdot\left(\frac{\nabla f}{\sqrt{|\nabla f|^2+\epsilon}}\right) = 0.$$

Refer to the argument in 9(b), a descent direction is:

$$-\frac{f-I}{\sqrt{(f-I)^2+\epsilon^2}}+\nabla\cdot\left(\frac{\nabla f}{\sqrt{|\nabla f|^2-\epsilon}}\right) \text{ for } (x,y)\in D,$$

and thus $E(f)$ can be iteratively minimized by updating f:

$$f^{n+1}(x,y)=f^n(x,y)-\tau\left[\frac{f-I}{\sqrt{(f-I)^2+\epsilon^2}}-\nabla\cdot\left(\frac{\nabla f}{\sqrt{|\nabla f|^2+\epsilon}}\right)\right] \text{ if } (x,y)\in D$$

for a small time step $\tau>0$.