MMAT 5390: Mathematical Imaging

Chapter 1: Basic concepts in Digital Image Processing

Mathematical imaging aims to develop mathematical models to process a digital image. The main tasks include enhancing the visual quality of a corrupted image and extracting important information from an image for the purpose of image understanding. Most mathematical models are done by transforming one image into another or by decomposing an image into meaningful components. In this chapter, we will explain some basic concepts in mathematical image processing. The definition of a digital image will firstly be described. The basic idea of image transformation and image decomposition will then be described in details. Finally, various measures to quantify the similarity between images will be explained.

1 Definition of digital images

A digital image captures the brightness at each pixel and represented by a numerical value, called the *pixel value*. Mathematically, a digital image can be understood as a matrix, which is array of numbers recording the pixel values at each pixels.

Definition 1.1. A digital **image** of width m pixels and height n pixels can be represented by a matrix $I \in \mathbb{R}^{n\times m}$. Mathematically, I belongs to the subset $\mathcal{I}$ of $n \times m$ matrices:

$$\mathcal{I} = \{I \in \mathbb{R}^{n\times m} \, : \, 0 \leq I(i,j) \leq R \text{ for } 1 \leq i \leq n, 1 \leq j \leq m\}$$

Typical values of the upper bound R of pixel values include 1 for greyscale images, and 255 for Red-Green-Blue (RGB) or Cyan-Magenta-Yellow-blacK (CMYK) images.

The main idea of mathematical imaging can be described as follows:

1. Given a noisy/distorted image $f \in \mathcal{I}$, find a suitable image transformation $T : \mathcal{I} \to \mathcal{I}$ such that $g := T(f)$ is the restored (good) image.
2. Given a distorted image $g \in \mathcal{I}$. We assume g is distorted by an image transformation $T : \mathcal{I} \to \mathcal{I}$ and corrupted by some noise n. Mathematically, we can write:

$$g = T(f) + n,$$

where f is the unknown good image. Given g and T, our goal is to find the good image f and n. This kind of problems is called the *inverse problem*. Mathematical imaging is often considered as an inverse problem.

2 Basic idea of image transformation

One important topic of image processing is to study how one image $f \in \mathcal{I}$ is transformed to another image $g \in \mathcal{I}$. Suppose g is obtained from f by an image transformation $\mathcal{O} : \mathcal{I} \to \mathcal{I}$. For simplicity, we consider $\mathcal{I} = M_{N\times N}(\mathbb{R})$ (the collection of all $N \times N$ matrices). We will first focus on the linear image transformation $\mathcal{O}$.

Definition 2.1. A image transformation $\mathcal{O} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ is **linear** if it satisfies:

$$\mathcal{O}(af + bg) = a\mathcal{O}(f) + b\mathcal{O}(g)$$

for all $f, g \in \mathcal{I}$ and $a, b \in \mathbb{R}$.

Example 1:

- Given $A \in M_{N\times N}(\mathbb{R})$, define $\mathcal{O}(f) = 2f + Af$ for all $f \in M_{N\times N}(\mathbb{R})$. $\mathcal{O}$ is a linear image transformation.
- Given $A \in M_{N\times N}(\mathbb{R})$ and $B \in M_{N\times N}(\mathbb{R})$, define $\mathcal{O}(f) = AfB$ for all $f \in M_{N\times N}(\mathbb{R})$. $\mathcal{O}$ is a linear image transformation.
- Define $\mathcal{O}(f) = fAf$ for all $f \in M_{N\times N}(\mathbb{R})$. $\mathcal{O}$ is NOT a linear image transformation.

Linear Image Transformation and Point Spread Function

Consider an image $f \in M_{N\times N}(\mathbb{R})$. Let

$$f = \begin{pmatrix} f(1,1) & f(1,2) & \cdots & f(1,N) \\ f(2,1) & f(2,2) & \cdots & f(2,N) \\ \vdots & \vdots & \ddots & \vdots \\ f(N,1) & f(N,2) & \cdots & f(N,N) \end{pmatrix} = \sum_{i=1}^{N}\sum_{j=1}^{N} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & f(i,j) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

where $f(i,j)$ is the intensity value at the i-th row and j-th column entry of f.
Let g be a transformed image from f by $\mathcal{O}$. In other words, $g = \mathcal{O}(f)$. Suppose $\mathcal{O}$ is linear. Then, the pixel value $g(\alpha,\beta)$ at the α-th row β-th column entry of g is given by:

$$g(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)h^{\alpha,\beta}(x,y)$$

where

$$h^{\alpha,\beta}(x,y) = \left[\mathcal{O}\left(\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}\right)\right]_{\alpha,\beta}$$

with 1 at x-th row and y-th column.

Remark. *$h^{\alpha,\beta}(x,y)$ determines how much the pixel value of f at (x,y) influences the pixel value of g at (α,β).*

Definition 2.2. $h^{\alpha,\beta}(x,y)$ is usually called the **point spread function (PSF)**.

Separable linear image transformation and Convolution

In mathematical imaging, two types of linear image transformations are particularly useful. They are, namely, the *separable linear image transformation* and *Convolution.*

Definition 2.3. An image transformation $\mathcal{O} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ is said to be a **separable** image transformation if there exists matrices $A \in M_{N\times N}(\mathbb{R})$ and $B \in M_{N\times N}(\mathbb{R})$ such that:

$$\mathcal{O}(f) = AfB$$

for all $f \in M_{N\times N}(\mathbb{R})$.

Example 2:

- An image transformation $\mathcal{O}$ that scales every pixel value by a positive constant α is a separable image transformation. In other words, $\mathcal{O}(f) = \alpha f = (\alpha\mathbf{I})f\mathbf{I}$, where $\mathbf{I}$ is the identity matrix.
- We will see that discrete Fourier transform (to be introduced later) is a separable image transformation.

Remark:

- We will see that a lot of image transformations that we use for image processing are separable image transformations.
- Discrete Haar transformation and discrete Fourier transformation (which will be discussed later) are commonly used separable image transformations in image processing.

Theorem 2.4. *Let $\mathcal{O}$ be a separable image transformation given by: $\mathcal{O}(f) = AfB$ for all $f \in M_{N\times N}(\mathbb{R})$, where $A \in M_{N\times N}(\mathbb{R})$ and $B \in M_{N\times N}(\mathbb{R})$. Then, the point spread function of $\mathcal{O}$ is given by:*

$$h^{\alpha,\beta}(x,y) = A(\alpha,x)B(y,\beta)$$

where $A(\alpha,x)$ is the (α,x) entry of A and $B(y,\beta)$ is the (y,β) entry of B.

Proof. Let $g = \mathcal{O}(f) = AfB$. The (α,β) entry of g is given by:

$$\begin{aligned} g(\alpha,\beta) &= \sum_{x=1}^{N} A(\alpha,x)(fB)(x,\beta) \\ &= \sum_{x=1}^{N} A(\alpha,x)\sum_{y=1}^{N} f(x,y)B(y,\beta) \\ &= \sum_{x=1}^{N}\sum_{y=1}^{N} A(\alpha,x)B(y,\beta)f(x,y). \end{aligned}$$

(where $(fB)(x,\beta)$ is the (x,β) entry of the matrix fB.)
Hence, the point spread function of $\mathcal{O}$ is given by: $h^{\alpha,\beta}(x,y) = A(\alpha,x)B(y,\beta)$. □

Remark: From the above theorem, we see that $\mathcal{O}$ is called separable as its point spread function can be written(or separated) as a product of two functions.

Definition 2.5. Consider two matrices $k \in M_{N\times N}(\mathbb{R})$ and $f \in M_{N\times N}(\mathbb{R})$. Assume that k and f are periodically extended, that is,

$$k(x,y) = k(x+pN, y+qN), f(x,y) = f(x+pN, y+qN)$$

where p and q are any integers. The **convolution** $k * f$ of k and f is a $N \times N$ matrix defined as

$$k * f(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} k(x,y)f(\alpha - x, \beta - y),$$

for $1 \leq \alpha, \beta \leq N$.

Example 3: Let $k = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $f = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Find $k * f \in M_{2\times 2}(\mathbb{R})$.
The (1,1) entry of $k * g$ is given by:

$$\begin{aligned} k * f(1,1) &= \sum_{x=1}^{2}\sum_{y=1}^{2} k(x,y)f(1-x,1-y) \\ &= k(1,1)f(0,0) + k(1,2)f(0,-1) + k(2,1)f(-1,0) + k(2,2)f(-1,-1) \\ &= k(1,1)f(2,2) + k(1,2)f(2,1) + k(2,1)f(1,2) + k(2,2)f(1,1) \text{ (by periodic extension)} \\ &= (1)(1) + (2)(1) + (3)(2) + (4)(1) = 13. \end{aligned}$$

Similarly, we can compute that $k * f(1,2) = 14$, $k * f(2,1) = 11$ and $k * f(2,2) = 12$.
Hence, $k * f = \begin{pmatrix} 13 & 14 \\ 11 & 12 \end{pmatrix}$.

Example 4: Let $k = \begin{pmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{pmatrix}$ and $f \in M_{3\times 3}(\mathbb{R})$. Find $k * f(2,2)$.

$$\begin{aligned} k * f(2,2) &= \sum_{x=1}^{3}\sum_{y=1}^{3} k(x,y) f(2-x, 2-y) \\ &= \frac{f(1,1)+f(1,2)+f(1,3)+f(2,1)+f(2,2)+f(2,3)+f(3,1)+f(3,2)+f(3,3)}{9}. \end{aligned}$$

It is just the average intensity values in the neighborhood of $f(2,2)$. In other words, taking average of the intensity values in the neighborhood can be expressed in terms of convolution. Taking average is a commonly used technique in image processing for image denoising.

Theorem 2.6. *Let $k \in M_{N\times N}(\mathbb{R})$. Define $\mathcal{O} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ by:*

$$\mathcal{O}(f) = k * f$$

for all $f \in M_{N\times N}(\mathbb{R})$. Then, $\mathcal{O}$ is a linear image transformation.

Proof. It follows immediately from the definition of convolution. □

Theorem 2.7. *Let $k \in M_{N\times N}(\mathbb{R})$ and $f \in M_{N\times N}(\mathbb{R})$. Then, $k * f = f * k$.*

Proof. Assuming that $\alpha, \beta > 1$,

$$\begin{aligned} k * f(\alpha,\beta) &= \sum_{x=1}^{N}\sum_{y=1}^{N} k(x,y) f(\alpha - x, \beta - y) \\ &= \sum_{\tilde{x}=\alpha-N}^{\alpha-1}\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y}) f(\tilde{x},\tilde{y}) \text{ (let } \tilde{x} = \alpha - x, \tilde{y} = \beta - y) \\ &= \sum_{\tilde{x}=\alpha-N}^{0} \left(\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-(\tilde{x}+N), \beta-\tilde{y}) f(\tilde{x}+N,\tilde{y}) \right) \\ &+ \sum_{\tilde{x}=1}^{\alpha-1} \left(\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y}) f(\tilde{x},\tilde{y}) \right) \text{ (by periodic extension)} \\ &= \sum_{\tilde{x}=\alpha}^{N} \left(\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y}) f(\tilde{x},\tilde{y}) \right) + \sum_{\tilde{x}=1}^{\alpha-1} \left(\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y}) f(\tilde{x},\tilde{y}) \right) \\ &= \sum_{\tilde{x}=1}^{N}\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y}) f(\tilde{x},\tilde{y}) \\ &= \sum_{\tilde{x}=1}^{N}\sum_{\tilde{y}=1}^{N} k(\alpha-\tilde{x}, \beta-\tilde{y}) f(\tilde{x},\tilde{y}) \\ &= f * k(\alpha,\beta). \end{aligned}$$

The case when $\alpha = 1$ or $\beta = 1$ can be shown similarly.

□

Definition 2.8. The point spread function $h^{\alpha,\beta}(x,y)$ of a linear image transformation is called **shift-invariant** if there exists a function $\tilde{h}$ such that

$$h^{\alpha,\beta}(x,y) = \tilde{h}(\alpha - x, \beta - y)$$

for all $1 \leq x, y, \alpha, \beta \leq N$.

Example 5: Let $\mathcal{O} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ be a linear image transformation defined by:

$$\mathcal{O}(f)(\alpha,\beta) = f(\alpha+1,\beta) + 2f(\alpha,\beta) - 2f(\alpha-1,\beta) + f(\alpha,\beta+1) - 2f(\alpha,\beta-1),$$

for all $1 \leq \alpha,\beta \leq N$ and $f \in M_{N\times N}(\mathbb{R})$. Show that $\mathcal{O}$ can be expressed in terms of a convolution. Suppose $\mathcal{O}(f) = k * f$ for some $k \in M_{N\times N}(\mathbb{R})$. Then,

$$\begin{aligned}
\mathcal{O}(f)(\alpha,\beta) &= k * f(\alpha,\beta) \\
&= \sum_{x=1}^{N}\sum_{y=1}^{N} k(x,y) f(\alpha-x,\beta-y) \\
&= ... + k(-1,0)f(\alpha+1,\beta) + k(0,0)f(\alpha,\beta) + k(1,0)f(\alpha-1,\beta) + k(0,-1)f(\alpha,\beta+1) \\
&+ k(0,1)f(\alpha,\beta-1) + ...
\end{aligned}$$

By setting $k(-1,0) = k(N-1,N) = 1$, $k(0,0) = k(N,N) = 2$, $k(1,0) = k(1,N) = -2$, $k(0,-1) = k(N,N-1) = 1$, $k(0,1) = k(N,1) = -2$ and set other entries of k equal to 0 otherwise. $\mathcal{O}(f)$ is then equal to $k * f$.
Again, we see that this weighted linear combination of intensity values of the neighborhood pixels can be expressed in terms of convolution. Convolution is commonly used in many imaging tasks.

Remark:

- Given $k \in M_{N\times N}(\mathbb{R})$. Let $\mathcal{O}$ be the linear image transformation defined by: $\mathcal{O}(f) = k * f$ for all $f \in M_{N\times N}(\mathbb{R})$. Then, the point spread function of $\mathcal{O}$ is shift-invariant.
- A lot of image processing tasks are related to convolution. For example, we will see the image blurring is related to convolution. As seen in Example 4, convolution is commonly used for image denoising. We will see that convolution can be used for feature extraction.

3 Similarity measure between images

In image processing, we often need to approximate an image by another image with better properties. For example, the main idea of image denoising (removing artifacts/noises from image) is to approximate an input noisy image by a 'smoother' image. In order to approximate an image, it is necessary to have a measure to quantify the similarity between different images.
Recall that a digital image can be considered as a matrix. To measure the similarity between two images, it is equivalent to defining a matrix norm. We first recall the definition of a vector norm.

Definition 3.1. A **vector norm** is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions:

1. $\|\mathbf{x}\| \geq 0$, $\|\mathbf{x}\| = 0$ only if $\mathbf{x} = \mathbf{0}$;
2. $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality);
3. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$;

for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

The most commonly used vector norms are the vector p-norms.

Definition 3.2. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}^n$, and let $p \geq 1$. The **vector p-norm** of $\mathbf{v}$, denoted by $\|\mathbf{v}\|_p$, is given by

$$\|\mathbf{v}\|_p := \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}}.$$

The limiting case for $p \to \infty$ is given by

$$\|\mathbf{v}\|_\infty := \lim_{p\to\infty} \|\mathbf{v}\|_p = \max_{1\leq i\leq n} |v_i|,$$

and is also called the supremum norm of $\mathbf{v}$.

One can check with Definition 3.1 to verify that the p-norms are indeed vector norms.
Having defined vector norms, a matrix norm can be induced from each vector norm.

Definition 3.3. Let $A \in \mathbb{R}^{n\times m}$ and $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ be a vector norm. We define the **induced matrix norm** $\|A\|$ to be the smallest $C \in \mathbb{R}$ such that

$$\|A\mathbf{x}\| \leq \mathbf{C}\|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathbb{R}^{\mathbf{m}},$$

or equivalently,

$$\|A\| = \sup_{\mathbf{x}\in\mathbb{R}^m, \mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \sup_{\mathbf{x}\in\mathbb{R}^m, \|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

However, not every matrix norm can be induced from a vector norm. In fact, matrix norms are defined in a similar manner to vector norms.

Definition 3.4. A **matrix norm** is a function $\|\cdot\| : \mathbb{R}^{n\times m} \to \mathbb{R}$ satisfying the following conditions:

1. $\|A\| \geq 0$, $\|A\| = 0$ only if $A = \mathbf{0}$;
2. $\|A + B\| \leq \|A\| + \|B\|$ (triangle inequality);
3. $\|\alpha A\| = |\alpha|\|A\|$;

for all $A, B \in \mathbb{R}^{n\times m}$ and $\alpha \in \mathbb{R}$.

For example, having defined the stacking operator and vector p-norms, another set of matrix p-norms can be defined as the vector p-norms of the stacked versions of matrices.

Definition 3.5. Let $A \in \mathbb{R}^{n\times m}$, and let $p \geq 1$. The **entrywise matrix p-norm** of A, denoted by $\|A\|_{p,e}$, is given by

$$\|A\|_{p,e} := \|\mathcal{S}A\|_p = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} |A(i,j)|^p\right)^{\frac{1}{p}}.$$

$\|A\|_{2,e}$ is also called the Frobenius norm (F-norm) of A; it is also denoted by $\|A\|_F$. Let $\mathbf{a}_j$ be the j-th column of A. We have

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} A(i,j)^2} = \sqrt{\sum_{j=1}^{m} \|\mathbf{a}_j\|_2^2} = \sqrt{\operatorname{tr}(A^T A)},$$

where $\operatorname{tr}(\cdot)$ is the trace of the matrix in the argument.
The limiting case for $p \to \infty$ is given by

$$\|A\|_\infty := \lim_{p\to\infty} \|A\|_p = \max_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} |A(i,j)|,$$

and is also called the entrywise supremum norm of A.

Remark. *In literature, the notations $\|A\|_p$ and $\|A\|_\infty$ are used for both the induced norms and the entrywise norms. Unless otherwise specified, in the notes we reserve these notations for induced norms, and denote the entrywise norms by $\|A\|_{p,e}$ and $\|A\|_{\infty,e}$.*

One may check with Definition 3.4 to verify that all induced norms and entrywise norms are indeed matrix norms.

Theorem 3.6. *The induced matrix 2-norm and the F-norm are invariant under multiplication by orthogonal matrices (an orthogonal matrix U satisfies $U^T U = UU^T = I$; its column vectors are orthonormal and its row vectors are also orthonormal), i.e. for any $A \in M_{n\times m}(\mathbb{R})$ and unitary $U \in M_{n\times n}(\mathbb{R})$, we have $\|UA\|_2 = \|A\|_2$ and $\|UA\|_F = \|A\|_F$.*

Proof. Since for any $x \in \mathbb{R}^m$,

$$\|UA\mathbf{x}\|_2^2 = (UA\mathbf{x})^T(UA\mathbf{x}) = \mathbf{x}^T A^T U^T U A\mathbf{x} = \mathbf{x}^T A^T A\mathbf{x} = \|A\mathbf{x}\|_2^2,$$

we have

$$\|UA\|_2 = \max_{\|\mathbf{x}\|_2=1} \|UA\mathbf{x}\|_2 = \max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2 = \|A\|_2.$$

Furthermore,

$$\|UA\|_F = \sqrt{\operatorname{tr}\big((UA)^T(UA)\big)} = \sqrt{\operatorname{tr}(A^T U^T U A)} = \sqrt{\operatorname{tr}(A^T A)} = \|A\|_F.$$

□

With matrix norms defined, we can measure the dissimilarity between two matrices (or images) by computing the norms of their difference matrix. Among the entrywise p-norms, the 1-norm and 2-norm are the most frequently used dissimilarity measures. The following figures demonstrate their different emphases.

Figure 1: The images on the left and on the right are equally similar to the image in the middle in terms of the entrywise 1-norm. On the other hand, the image on the right is significant less similar to the image in the middle in terms of the entrywise 2-norm than the image on the left.

Figure 2: The images on the left and on the right are equally similar to the image in the middle in terms of the entrywise 2-norm. On the other hand, the image on the right is significant less similar to the image in the middle in terms of the entrywise 1-norm than the image on the left.

As seen from the figures, the 1-norm is more sensitive to widespread deviation in large regions, whereas the 2-norm is more sensitive to extreme pixel value differences, even if they are restricted to small regions. This trend goes on across different values of $p \geq 1$.

4 Transformation matrix of linear image transformation

A linear image transformation $\mathcal{O} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ is a linear transformation from $M_{N\times N}(\mathbb{R})$ to itself. $M_{N\times N}(\mathbb{R})$ is a N^2-dimensional vector space. From linear algebra, a linear transformation from a N^2-dimensional vector space to itself can be represented by a N^2 by N^2 matrix, called the *transformation matrix*. Hence, $\mathcal{O}$ can be represented by a transformation matrix $H \in M_{N^2\times N^2}(\mathbb{R})$.
To understand the idea, let's consider a deformed image $g = \mathcal{O}(f)$, where $f, g \in M_{N\times N}(\mathbb{R})$. In other words, the image g is obtained from f by transfoming f by $\mathcal{O}$.

Then, we have the following equation (*):

$$\begin{aligned} g(\alpha,\beta) &= f(1,1)h^{\alpha,\beta}(1,1) + f(2,1)h^{\alpha,\beta}(2,1) + \cdots + f(N,1)h^{\alpha,\beta}(N,1) \\ &\quad + f(1,2)h^{\alpha,\beta}(1,2) + \cdots + f(N,2)h^{\alpha,\beta}(N,2) + \cdots \\ &\quad + f(1,N)h^{\alpha,\beta}(1,N) + \cdots + f(N,N)h^{\alpha,\beta}(N,N) \end{aligned}$$

Let $\vec{f} = \begin{pmatrix} f(1,1) \\ \vdots \\ f(N,1) \\ f(1,2) \\ \vdots \\ f(N,2) \\ \vdots \\ f(N,N) \end{pmatrix}$ and $\vec{g} = \begin{pmatrix} g(1,1) \\ \vdots \\ g(N,1) \\ g(1,2) \\ \vdots \\ g(N,2) \\ \vdots \\ g(N,N) \end{pmatrix}$. $\vec{f}$ and $\vec{g}$ are called the vectorized images of f and g respectively.

Then, $\vec{g} = H\vec{f}$. The entries of H can be determined by equation (*), which contains $h^{\alpha,\beta}(i,j)'s$. The following example can help us to understand the idea.

Example 4.1. A linear image transformation is such that it replaces the value of each pixel by the average of its four nearest neighbours. Assume the image is repeated in all directions (periodically extended). Apply this operator $\mathcal{O}$ to a 3×3 image. Find the transformation matrix corresponding to $\mathcal{O}$.

Solution. The 3×3 image looks like

$$\begin{array}{c:ccc:c} f_{33} & f_{31} & f_{32} & f_{33} & f_{31} \\ \hdashline f_{13} & f_{11} & f_{12} & f_{13} & f_{11} \\ f_{23} & f_{21} & f_{22} & f_{23} & f_{21} \\ f_{33} & f_{31} & f_{32} & f_{33} & f_{31} \\ \hdashline f_{13} & f_{11} & f_{12} & f_{13} & f_{11} \end{array}$$

$$\therefore g_{11} = \frac{f_{21} + f_{31} + f_{12} + f_{13}}{4}$$

$$g_{21} = \frac{f_{11} + f_{31} + f_{22} + f_{23}}{4},$$

and so on.
By a simple checking, we observe that the transformation matrix H can be written as

$$H = \begin{bmatrix} 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 0 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 1/4 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 & 0 & 0 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 0 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 0 & 0 & 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 \end{bmatrix}.$$

Remark: Determining the transformation matrix of a linear image transformation is needed when we implement the mathematical models using MATLAB.