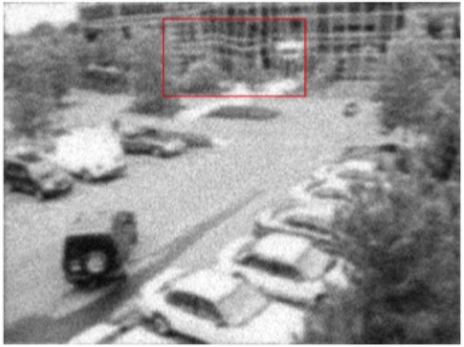
Lecture 9: Image deblurring



Atmospheric turbulence

Motion Blur

Speeding problem

Image deblurring in the frequency domain:

Mathematical formulation of image blurring

(not a matrix, just a transformation)

$$f \xrightarrow{D} g$$

clean, original image

blurry image of f

Let g be the observed (blurry) image.

Let f be the original (good) image.

Model g as: $g = D(f) + n$

where D is the degradation function/operator and n is the additive noise.

Assumption on D:

1. D is position invariant:

   Let $g(x,y) = D(f)(x,y)$ and let $\tilde{f}(x,y) := f(x-\alpha, y-\beta)$.

   Then: $D(\tilde{f})(x,y) = g(x-\alpha, y-\beta) = D(f)(x-\alpha, y-\beta)$

2. Linear: $D(f_1 + f_2) = D(f_1) + D(f_2)$

   $D(\alpha f) = \alpha D(f)$ where α is a scalar multiplication.

With the above assumption, consider an impluse image $\delta \in M_{(N+1)\times(N+1)}$ (indices taken between $-\frac{N}{2}$ to $\frac{N}{2}$)

$$\delta(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0) \\ 0 & \text{if } (x,y) \neq (0,0) \end{cases}$$

Let $\tilde{\delta}_{\alpha,\beta}$ be the translated image of δ by (α,β):

$$\tilde{\delta}_{\alpha,\beta}(x,y) = \delta(x-\alpha, y-\beta) \quad \text{for} \quad -\frac{N}{2} \leq x, y \leq \frac{N}{2}$$

Note: $f(x,y) = f * \delta(x,y) = \sum_{\alpha=-\frac{N}{2}}^{N/2-1} \sum_{\beta=-N/2}^{N/2-1} f(\alpha,\beta)\,\delta(x-\alpha, y-\beta) = \sum_{\alpha=-\frac{N}{2}}^{\frac{N}{2}} \sum_{\beta=-\frac{N}{2}}^{\frac{N}{2}} f(\alpha,\beta)\,\tilde{\delta}_{\alpha,\beta}(x,y)$

for all $-\frac{N}{2} \leq x, y \leq \frac{N}{2}$.

$$\therefore \quad f = \sum_{\alpha=-\frac{N}{2}}^{\frac{N}{2}} \sum_{\beta=-\frac{N}{2}}^{\frac{N}{2}} \underbrace{f(\alpha,\beta)}_{\in \mathbb{R}} \underbrace{\tilde{\delta}_{\alpha,\beta}}_{M_{(N+1)\times(N+1)}}$$

Let g be the blurry image of f. That is, $g = D(f)$

$$g = D(f) = D\left(\sum_{\alpha=-\frac{N}{2}}^{\frac{N}{2}} \sum_{\beta=-\frac{N}{2}}^{\frac{N}{2}} f(\alpha,\beta)\, \tilde{\delta}_{\alpha,\beta}\right)$$

$$= \sum_{\alpha=-\frac{N}{2}}^{\frac{N}{2}} \sum_{\beta=-\frac{N}{2}}^{\frac{N}{2}} f(\alpha,\beta)\, D(\tilde{\delta}_{\alpha,\beta}) \quad \text{(linearity of } D\text{)}$$

$$\therefore \quad g(x,y) = \sum_{\alpha=-\frac{N}{2}}^{\frac{N}{2}} \sum_{\beta=-\frac{N}{2}}^{\frac{N}{2}} f(\alpha,\beta)\, D(\tilde{\delta}_{\alpha,\beta})(x,y)$$

$$= \sum_{\alpha=-\frac{N}{2}}^{\frac{N}{2}} \sum_{\beta=-\frac{N}{2}}^{\frac{N}{2}} f(\alpha,\beta)\, D(\delta)(x-\alpha, y-\beta) \quad \text{(position-invariant)}$$

$$= f * h(x,y) \quad \text{where} \quad h = D(\delta)$$

$$\therefore \quad g = f * h$$

$\therefore$ With the above assumption,

Degradation / Blur = Convolution

Remark:

- $g(x,y) = h * f(x,y)$

In the frequency domain,

$$G(u,v) = c\, H(u,v)\, F(u,v)$$

constant

$\therefore$ Deblurring can be done by:

Compute: $F(u,v) \approx \dfrac{G(u,v)}{c H(u,v)}$

$G(u,v)$ — from observe

$c H(u,v)$ — from known degradation

Obtain: $f(x,y) = DFT^{-1}(F(u,v))$

Examples of degradation function $H(u,v)$

1. Atmospheric turbulence blur:

$$H(u,v) = e^{-k(u^2+v^2)^{5/6}}$$

where k = degree of turbulence

$k = 0.0025$ (severe)

$k = 0.001$ (mild)

$k = 0.00025$ (low turbulence)

2. Out of focus blur:

In the frequency domain, define $H(u,v)$ as the DFT of

$$h(x,y) = \begin{cases} 1 & \text{if } x^2+y^2 \le D_0^2 \\ 0 & \text{otherwise} \end{cases}$$

$c\, H \odot \hat{I}$

$\updownarrow$

$h * I$

3. Uniform Linear Motion Blur

Assume $f(x,y)$ (original) undergoes planar motion during acquisition.

Let $(x_0(t), y_0(t))$ be the motion components (displacements) of the scene in the x- and y-directions
(t: time)

Let T be the total exposure time.

The observed image is given by:

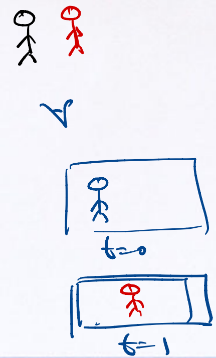
$$g(x,y) = \int_0^T f(x - x_0(t),\, y - y_0(t))\, dt$$

Now, let $G(u,v) = \mathrm{DFT}(g)(u,v)$, then:

$$G(u,v) = \frac{1}{N^2}\sum_x \sum_y g(x,y)\, e^{-j\frac{2\pi}{N}(ux+vy)}$$

$$= \frac{1}{N^2}\sum_x \sum_y \int_0^T f(x - x_0(t),\, y - y_0(t))\, dt\; e^{-j\frac{2\pi}{N}(ux+vy)}$$

$$= \int_0^T \left(\sum_x \sum_y f(x - x_0(t),\, y - y_0(t))\, e^{-j\frac{2\pi}{N}(ux+vy)} \right) dt$$



Recall that $DFT(f(x-x_0, y-y_0)) = F(u,v)\, e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))}$.

$F = DFT(f)$

We have: $$G(u,v) = \int_0^T \left[F(u,v)\, e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))} \right] dt$$

$$= F(u,v) \int_0^T e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))}\, dt$$

$$= F(u,v)\, H(u,v)$$

$\therefore$ Degradation function in the frequency domain is given by:

$$H(u,v) = \int_0^T e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))}\, dt$$

(Speeding problem !!)

Example: Suppose the camera is moving left horizontally with a constant speed c.

That is, the image at time t is given by:

$$I^*(x,y) = I(x, y-ct)$$

Then: the degradation function is given by:

$$H(u,v) = \int_0^T e^{-j\frac{2\pi}{N}(v(ct))}\,dt$$

Remark: Once the degradation function is known, the original image can be restored by: $\mathrm{IDFT}\left(\frac{G(u,v)}{H(u,v)}\right)$ (given that there's no noise)

What if there is noise??

Image deblurring in the frequency domain: (Assume H is known)

Method 1: Direct inverse filtering

Let $T(u,v) = \dfrac{1}{cH(u,v) + \underbrace{\varepsilon \, \mathrm{sgn}(H(u,v))}_{\text{Avoid singularity}}}$ ($\mathrm{sgn}(z) = 1$ if $\mathrm{Re}(z) \geq 0$ and $\mathrm{sgn}(z) = -1$ otherwise)

Compute $\hat{F}(u,v) = G(u,v)\, T(u,v)$.

Find inverse DFT of $\hat{F}(u,v)$ to get an image $\hat{f}(x,y)$.

Good: Simple

Bad: Boast up noise

$$\hat{F}(u,v) = \underset{\substack{\parallel \\ cH(u,v)F(u,v) + N(u,v)}}{G(u,v)}\, T(u,v) \approx F(u,v) + \frac{N(u,v)}{cH(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$$

Note: $H(u,v)$ is big for (u,v) close to $(0,0)$ (Keep low frequencies)

is small for (u,v) far away from $(0,0)$

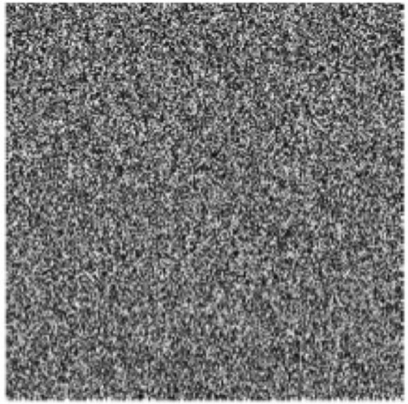
$\therefore \dfrac{N(u,v)}{cH(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$ is big for (u,v) far away from $(0,0)$ — Large gain in high frequencies

$\Downarrow$

Boast up noises!!

Original

Blurred image



Direct inverse filtering

Method 2: Modified inverse filtering

Let $B(u,v) = \dfrac{1}{1+\left(\frac{u^2+v^2}{D^2}\right)^n}$ and $T(u,v) = \dfrac{B(u,v)}{c\,H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$.

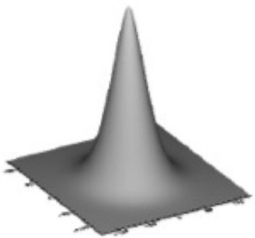
Then define: $\hat{F}(u,v) = T(u,v)\,G(u,v) \approx F(u,v)\,B(u,v) + \dfrac{N(u,v)\,B(u,v)}{c\,H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$

$\dfrac{N(u,v)\,B(u,v)}{c\,H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))} \approx \dfrac{N(u,v)}{c\,H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$ for $(u,v) \approx (0,0)$

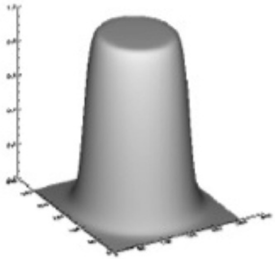
$\dfrac{N(u,v)\,B(u,v)}{c\,H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$ is small (as $B(u,v)$ is small) for (u,v) far away from $(0,0)$.

$\dfrac{B(u,v)}{c\,H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$ suppresses the high-frequency gain.

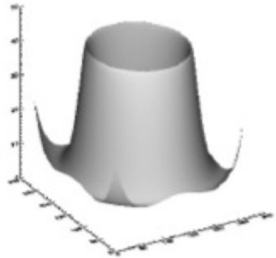
Bad: Has to choose D and n carefully.



$H(u,v)$



$B(u,v)$: $D = 90, n = 8$



Inverse B/H

Original Image $G(u,v)$

Blurred using $D = 90, n = 8$

Restored with a best D and n.

Method 3: Wiener filter

Let $T(u,v) = \dfrac{\overline{cH(u,v)}}{|cH(u,v)|^2 + \dfrac{S_n(u,v)}{S_f(u,v)}}$ where $S_n(u,v) = |N(u,v)|^2$

$S_f(u,v) = |F(u,v)|^2$

If $S_n(u,v)$ and $S_f(u,v)$ are not known, then we let $K = \dfrac{S_n(u,v)}{S_f(u,v)}$ to get:

$$T(u,v) = \frac{\overline{cH(u,v)}}{|cH(u,v)|^2 + K}$$

Let $\hat{F}(u,v) = T(u,v)\,G(u,v)$. Compute $\hat{f}(x,y) =$ inverse DFT of $\hat{F}(u,v)$.

In fact, the Wiener filter can be described as an inverse filtering as follows:

$$\hat{F}(u,v) = \left[\left(\frac{1}{cH(u,v)}\right)\left(\frac{|cH(u,v)|^2}{|cH(u,v)|^2 + K}\right)\right] G(u,v)$$

Behave like "Modified inverse filtering"

≈ 0 if $H(u,v) \approx 0$ (if (u,v) far away from 0)

≈ 1 if $H(u,v)$ is large (if $(u,v) \approx (0,0)$)

What does Wiener filtering do mathematically?

We can show: Wiener filter minimizes the mean square error:

$$\varepsilon^2(\underset{\substack{\uparrow \\ \text{original}}}{f}, \underset{\substack{\uparrow \\ \text{Restored}}}{\hat{f}}) = \sum_{x=-\frac{N}{2}}^{\frac{N}{2}} \sum_{y=-\frac{N}{2}}^{\frac{N}{2}} \left| f(x,y) - \hat{f}(x,y) \right|^2 = \| f - \hat{f} \|_F^2$$

Next time!