Lecture 5:

Haar transformation

(From now on, we assume all images $f = (f_{ij})_{\substack{0 \le i \le N-1 \\ 0 \le j \le N-1}}$

Definition: (Haar functions) The Haar functions are defined as follows

$$H_0(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

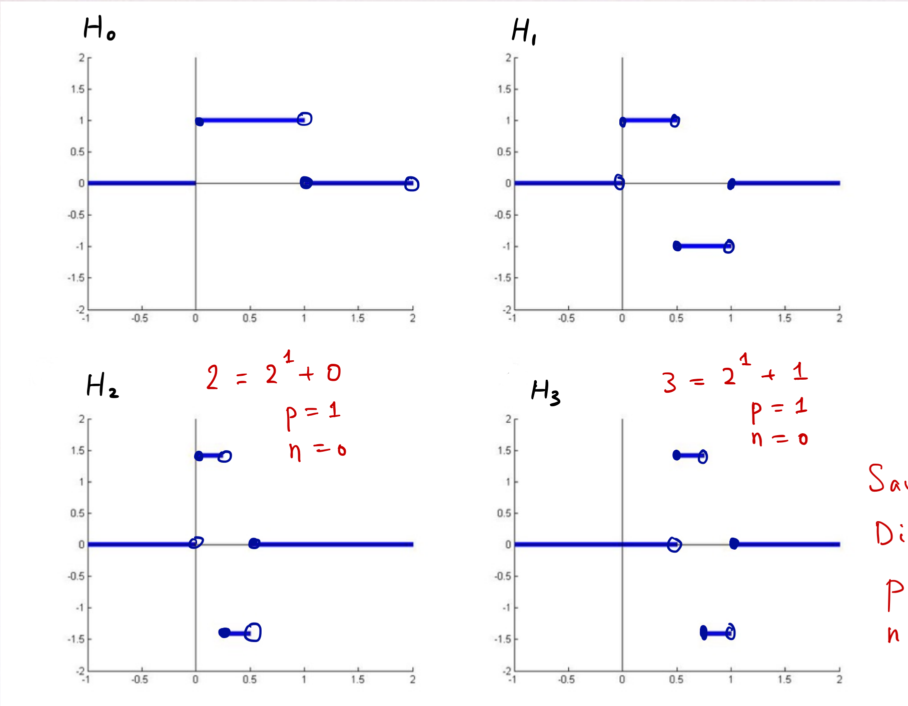
$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$H_{2^p+n} \equiv \begin{cases} \sqrt{2}^p & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^p & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p = 1, 2, \ldots, n = 0, 1, 2, \ldots, 2^p - 1$

Remark: If p is larger, H_{2^p+n} is compactly supported region

Examples of Haar functions:



Same wavefront

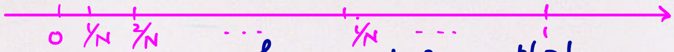
Different locations

$p \leftrightarrow$ wavefront

$n \leftrightarrow$ location

Definition (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is done by dividing $[0,1]$ into partitions



Let $H(k,l) \equiv H_k\left(\frac{l}{N}\right)$ where $k, l = 0, 1, 2, \ldots, N-1$

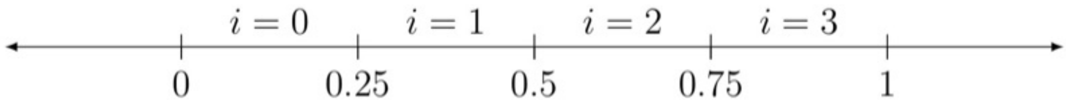
We obtain the Haar Transform matrix $\tilde{H} \equiv \frac{1}{\sqrt{N}} H$ where $H \equiv \left(H(k,l)\right)_{0 \le k, l \le N-1}$

The Haar Transform of $f \in M_{n \times n}$ is defined as

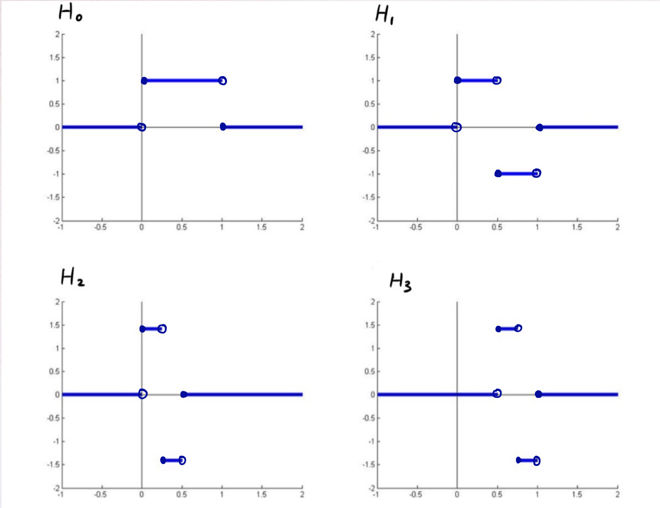
$$g = \tilde{H} f \tilde{H}^T$$

Example Compute the Haar Transform matrix for a 4×4 image.

Solution: Divide $[0,1]$ into 4 portions:



Need to check:



$$\begin{pmatrix} H(0,0) & H(0,1) & H(0,2) & H(0,3) \\ H(1,0) & H(1,1) & H(1,2) & H(1,3) \\ H(2,0) & H(2,1) & H(2,2) & H(2,3) \\ H(3,0) & H(3,1) & H(3,2) & H(3,3) \end{pmatrix}$$

$\parallel$

We get that:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \quad \text{and} \quad \tilde{H} = \frac{1}{\sqrt{4}} H = \frac{1}{2} H$$

Easy to check that $\tilde{H}^T \tilde{H} = I$.

$\parallel$

$$\begin{pmatrix} H_0(\frac{0}{4}) & H_0(\frac{1}{4}) & H_0(\frac{2}{4}) & H_0(\frac{3}{4}) \\ H_1(\frac{0}{4}) & H_1(\frac{1}{4}) & H_1(\frac{2}{4}) & H_1(\frac{3}{4}) \\ & & & \\ & & & \end{pmatrix}$$

Example 2 Compute the Haar Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

$$g = \tilde{H} f \tilde{H}^T = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

More zeros

Example 3 Suppose g in Example 2 is changed to:

$$g = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Reconstruct the original image.

Solution:

$$f = \tilde{H}^T g \tilde{H} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Localized error

Remark:

1. Haar Transform usually produces coefficient matrix with more zeros!

2. Localized error in coefficient matrix causes localized error in the reconstructed image

Elementary images under Haar transform:

Using Haar transform, f can be written as:

$$f = \tilde{H}^T g \tilde{H}$$

(g ← transformed image)

Let $\tilde{H} = \begin{pmatrix} - \vec{h}_0^T - \\ - \vec{h}_1^T - \\ \vdots \\ - \vec{h}_{N-1}^T - \end{pmatrix}$. Then: $f = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} g_{ij} \vec{h}_i \vec{h}_j^T$

($\vec{h}_i \vec{h}_j^T = I_{ij}^H$)

I_{ij}^T = elementary images under Haar Transform.

e.g. For 8×8 images, we have $64 = 8 \times 8$ elementary images:

$$\begin{matrix} \vec{h}_0 \vec{h}_0^T & \vec{h}_0 \vec{h}_1^T & \cdots & \vec{h}_0 \vec{h}_7^T \\ \vec{h}_1 \vec{h}_0^T & \vec{h}_1 \vec{h}_1^T & \cdots & \vec{h}_1 \vec{h}_7^T \\ \vdots & \vdots & & \vdots \\ \vec{h}_7 \vec{h}_0^T & \vec{h}_7 \vec{h}_1^T & & \vec{h}_7 \vec{h}_7^T \end{matrix}$$

(each $\vec{h}_i \vec{h}_j^T \in M_{8\times 8}(\mathbb{R})$)

Recall:

- Image decomposition f

$$f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{I_{ij}}_{\text{elementary images}}$$

① Storage saving

② Image processing by modifying transformed image (coefficient matrix)
(e.g. Removing coefficients associated to high-frequency elementary images)

- 2 Separable Image Transformation:

① SVD (elementary images not universal and meaningless)

② Haar (elementary images universal and meaningful) — unsmooth

Discrete Fourier Transform:

Definition:

The 2D DFT of a $M \times N$ image $g = (g(k,l))_{k,l}$, where $0 \le k \le M-1$, $0 \le l \le N-1$ is defined as:

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

(where $j = \sqrt{-1}$, $e^{j\theta} = \cos\theta + j\sin\theta$)

Remark: The inverse of DFT is given by:

$$g(p,q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{g}(m,n)\, e^{j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)}$$

(no $\frac{1}{Mn}$!) — DFT of g — (no -ve sign)

Proof of Inverse DFT:

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{pm}{M}+\frac{qn}{N}\right)}\hat{g}(m,n) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{pm}{M}+\frac{qn}{N}\right)}\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M}+\frac{ln}{N}\right)}$$

$$= \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g(k,l)\, e^{j2\pi\left(\frac{(p-k)m}{M}+\frac{(q-l)n}{N}\right)}$$

$$= \underbrace{\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\sum_{m=0}^{M-1} e^{j2\pi\left(\frac{(p-k)m}{M}\right)}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{(q-l)n}{N}\right)}}_{(*)}$$

Note that: $$\sum_{m=0}^{M-1} e^{j2\pi\left(\frac{mt}{M}\right)} = \underbrace{\frac{\left[e^{j2\pi\left(\frac{t}{M}\right)}\right]^{M}-1}{e^{j2\pi\left(\frac{t}{M}\right)}-1}}_{\text{if } t\neq 0} = M\,\delta(t) := \begin{cases} M & t=0 \\ 0 & t\neq 0 \end{cases}$$

$\therefore$ (*) becomes: $$\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\, M\,\delta(p-k)\, N\,\delta(q-l) = g(p,q).$$

DFT in Matrix form

Theorem: Consider a $N \times N$ image g, the DFT of g can be written as:

$$\hat{g} = \mathcal{U} g \mathcal{U} \quad \text{(DFT in matrix form)}$$

where $\mathcal{U} = (U_{kl})_{0 \le k, l \le N-1} \in M_{N \times N}$ and $\mathcal{U}_{kl} = \frac{1}{N} e^{-j \frac{2\pi kl}{N}}$.

Proof: Need to check $\hat{g}(k,l) = (\mathcal{U} g \mathcal{U})(k,l)$ (k-th row, l-th col of $\mathcal{U} g \mathcal{U}$)

$$\text{LHS} = \hat{g}(k,l) = \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} g(m,n) e^{-j 2\pi \left(\frac{km}{N} + \frac{ln}{N}\right)}$$

$$\text{RHS}: \mathcal{U} g \mathcal{U} = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} g(m,n) \begin{pmatrix} | \\ \vec{u}_m \\ | \end{pmatrix} \left(- \vec{u}_n - \right)$$

$$\mathcal{U} = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} - \vec{u}_1 - \\ - \vec{u}_2 - \\ \vdots \\ - \vec{u}_N - \end{pmatrix}$$

$$\vec{u}_m = \begin{pmatrix} u_{0m} \\ u_{1m} \\ \vdots \\ u_{N-1\,m} \end{pmatrix} = \frac{1}{N} \begin{pmatrix} e^{-j\frac{2\pi(0)m}{N}} \\ \vdots \\ e^{-j\frac{2\pi(k)m}{N}} \\ \vdots \\ e^{-j\frac{2\pi(N-1)m}{N}} \end{pmatrix}$$

$$\mathcal{U} g \mathcal{U}(k,l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} g(m,n) \frac{e^{-j 2\pi \frac{km}{N}}}{N} \cdot \frac{e^{-j 2\pi \frac{ln}{N}}}{N}$$

$$= \text{LHS}$$

$$\vec{u}_n = (u_{n0}, u_{n1}, \ldots, u_{nN}) = \frac{1}{N} \left(e^{-j\frac{2\pi(0)(n)}{N}}, \ldots, e^{-j\frac{2\pi(l)(n)}{N}}, \ldots \right)$$

Theorem: $U^* U = \frac{1}{N} I$ where $U^* = (\overline{U})^T$ (conjugate transpose)

$(\overline{a + jb} = a - jb)$

$(\overline{e^{j\theta}} = \overline{\cos\theta + j\sin\theta} = \cos\theta - j\sin\theta = e^{-j\theta})$

$U U^* = \frac{1}{N} I.$

$\therefore U^{-1} = (NU)^*$

Proof: Consider $(U^* U)(k, l)$ (k-th row, l-th col of U^*U)

$(U^* U)(k,l) = (\overline{\text{k-th row of } U^*}) \, (|)$ l-col of U

Let $U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}$

$\overline{U} = \begin{pmatrix} — \vec{u}_1^T — \\ — \vec{u}_2^T — \\ \vdots \\ — \vec{u}_N^T — \end{pmatrix}$

$(\bar{U})^* = \begin{pmatrix} — (\overline{\vec{u}_1^T}) — \\ \vdots \\ — (\overline{\vec{u}_N^T}) — \end{pmatrix}$

$= \overline{(\vec{u}_k^T)} \, \vec{u}_l$

$= \left(\overline{\frac{e^{-j2\pi\frac{k(0)}{N}}}{N}}, \dots, \overline{\frac{e^{-j2\pi\frac{k\alpha}{N}}}{N}}, \dots, \overline{\frac{e^{-j2\pi\frac{k(N-1)}{N}}}{N}}\right) \begin{pmatrix} \frac{e^{-j2\pi\frac{l(0)}{N}}}{N} \\ \vdots \\ \frac{e^{-j2\pi\frac{l(\alpha)}{N}}}{N} \\ \vdots \\ \frac{e^{-j2\pi\frac{l(N-1)}{N}}}{N} \end{pmatrix}$

$= \sum_{\alpha=0}^{N-1} \frac{e^{j2\pi\frac{k\alpha}{N}}}{N} \frac{e^{-j2\pi\frac{l\alpha}{N}}}{N} = \sum_{\alpha=0}^{N-1} \frac{e^{-j2\pi(\alpha)(l-k)}}{N^2}$

$= \frac{1}{N}\delta(l-k)$

$$\therefore \quad U^* U(k,l) = \begin{cases} \frac{1}{N} & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

$$\Rightarrow \quad U^* U = \frac{1}{N} I$$

Similarly, $UU^* = \frac{1}{N} I$

Image decomposition by DFT

Suppose $\hat{g} = DFT(g) = \mathcal{U} g \mathcal{U}$

Then: $\mathcal{U}\mathcal{U}^* = \frac{1}{N} I = \mathcal{U}^*\mathcal{U}$

$\therefore \quad g = (N\mathcal{U})^* \hat{g} (N\mathcal{U})^*$

$\therefore \quad g = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{g}_{kl} \vec{w}_k \vec{w}_l^T$ ← Elementary image of DFT

where $\vec{w}_k = k^{th}$ col of $(N\mathcal{U})^*$