Lecture 4:

Recall:

SVD

$A \in M_{M\times N}(\mathbb{R})$

$$A = U \Sigma V^T$$

$U \in M_{M\times M}$, $\Sigma \in M_{M\times N}$, $V^T \in M_{N\times N}$

U = orthogonal $(U^T U = I)$

V = orthogonal $(V^T V = I)$

Σ = "diagonal" if $\Sigma_{ij} = 0$ for $i \neq j$

If $\underset{\uparrow \atop M_{N\times N}}{I} = U \Sigma V^T$ where $U = \left(\vec{u}_1 \; \vec{u}_2 \; \cdots \; \vec{u}_N \right)$

$$V = \left(\vec{v}_1 \; \vec{v}_2 \; \cdots \; \vec{v}_N \right)$$

then :

$$I = \left(\vec{u}_1 \; \cdots \; \vec{u}_N \right) \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_N \end{pmatrix} \begin{pmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_N^T \end{pmatrix}$$

$$I = \sum_{i=1}^{N} \sigma_i \underbrace{\boxed{\vec{u}_i \, \vec{v}_i^T}}_{\text{elementary image under SVD}}$$

Suppose $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

$\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_N = 0$

Then: $I = \sum_{i=1}^{r} \sigma_i \, \vec{u}_i \, \vec{v}_i^T$

How to compute SVD

Let $A \in M_{n\times n}$

Step 1: Find eigenvalues $\{\overset{\sigma_1^2}{\overset{\shortparallel}{\lambda_1}}, \overset{\sigma_2^2}{\overset{\shortparallel}{\lambda_2}}, \dots, \overset{\sigma_n^2}{\overset{\shortparallel}{\lambda_n}}\}$ and orthonormal eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of $A^TA \in M_{n\times n}$ (with $\|\vec{v}_j\| = 1$, $j = 1, \dots, n$)

[Recall: $(A^TA)\vec{v}_j = \lambda_j \vec{v}_j$]

Step 2: Define:

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix} \in M_{n\times n}$$

Step 3: For non-zero $\sigma_1, \sigma_2, \dots, \sigma_r$,

let $\vec{u}_1 = \dfrac{A\vec{v}_1}{\sigma_1}$, $\vec{u}_2 = \dfrac{A\vec{v}_2}{\sigma_2}$, $\dots$, $\vec{u}_r = \dfrac{A\vec{v}_r}{\sigma_r}$

Step 4: Extend $\{\vec{u}_1, \dots, \vec{u}_r\}$ to the basis $\{\vec{u}_1, \dots, \vec{u}_r, \dots, \vec{u}_n\}$ of $\mathbb{R}^n$

Step 5: Let:

$$U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \\ | & | & & | \end{pmatrix} \in M_{m\times m}$$

$$V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{n\times n}$$

Then: $A = U\Sigma V^T$

Example 2.1: Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

We have

$$A^T A = \begin{pmatrix} 9 & 8 \\ 8 & 9 \end{pmatrix}.$$

Now, eig(A^*A) are 17 and 1, and so $\sigma_1 = \sqrt{17}$, $\sigma_2 = 1$ and

$$\tilde{\Sigma} = \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \Sigma = \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Moreover,

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$u_i = \frac{A\vec{v}_i}{\sigma_i}$$

Since

$$\sigma_1 \vec{u}_1 = A\vec{v}_1,$$

$$\frac{A\vec{v}_1}{\sigma_1} =$$

we have

$$\vec{u}_1 = \frac{1}{\sqrt{17}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{34}} \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}.$$

$$\frac{A\vec{v}_2}{\sigma_2} =$$

Similarly, we have

$$\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The matrix U is, therefore, given by

$$U = \begin{array}{c} \\ \\ \end{array} \begin{array}{c} \vec{u}_1 = \quad \vec{u}_2 = \\ \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \\ \frac{4}{\sqrt{34}} & 0 & \mathbf{u}_3 \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \end{pmatrix} \end{array}$$

for some vector $\mathbf{u}_3$ orthonormal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. One possibility is

$$(\text{cross-product}) \quad \vec{u}_3 = \frac{1}{\sqrt{17}} \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}.$$

Finally, the SVD of A is given by

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \\ \frac{4}{\sqrt{34}} & 0 & \frac{-3}{\sqrt{17}} \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \end{pmatrix} \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}.$$

Definition: For any k $(0 \le k \le r)$, we define

$$g_k = \sum_{j=1}^{k} \sigma_j \vec{u}_j \vec{v}_j^{\,T} \quad (\text{rank-}k \text{ approximation of } g)$$

$$\| \\ \underbrace{U \begin{pmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_k & \\ & & & & \ddots \end{pmatrix} V^T}_{\text{Rank-}k\,!!}$$

Error of the approximation by SVD

Theorem: Let $f = \sum_{j=1}^{r} \sigma_j \vec{u}_j \vec{v}_j^T$ be the SVD of a $M \times N$ image f. For any $k < r$, and $f_k = \sum_{j=1}^{k} \sigma_j \vec{u}_j \vec{v}_j^T$, we have: $\| f - f_k \|_F^2 = \sum_{i=k+1}^{r} \underbrace{\sigma_i^2}_{}$ $\quad \lambda_i$

$$\begin{pmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iM} \end{pmatrix} \begin{pmatrix} v_{i1} & \cdots & v_{iN} \end{pmatrix} = \begin{pmatrix} u_{i1}v_{i1} & u_{i1}v_{i2} & \cdots \\ & u_{im}v_{in} & \\ & & \end{pmatrix}$$

Proof: Let $f = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$.

Let $D \equiv f - f_k = \sum_{i=k+1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T \in M_{M \times N}$.

Then, the m-th row, n-th col entry of D is given by:

$$D_{mn} = \sum_{i=k+1}^{r} \sigma_i u_{im} v_{in} \in \mathbb{R} \quad \text{where} \quad \vec{u}_i = \begin{pmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iM} \end{pmatrix} ; \quad \vec{v}_i = \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{iN} \end{pmatrix}$$

$$\therefore D_{mn}^2 = \left(\sum_{i=k+1}^{r} \sigma_i u_{im} v_{in} \right)^2 = \sum_{i=k+1}^{r} \sigma_i^2 u_{im}^2 v_{in}^2 + \sum_{i=k+1}^{r} \sum_{\substack{j=k+1 \\ j \neq i}}^{r} \sigma_i \sigma_j u_{im} v_{in} u_{jm} v_{jn}.$$

Thus, $\|D\|_F^2 = \sum_m \sum_n D_{mn}^2$

$$= \sum_m \sum_n \left(\sum_{i=k+1}^{r} \sigma_i^2 \, u_{im}^2 \, v_{in}^2 + \sum_{i=k+1}^{r} \sum_{\substack{j=k+1 \\ j \neq i}}^{r} \sigma_i \sigma_j \, u_{im} v_{in} u_{jm} v_{jn} \right)$$

$$= \sum_{i=k+1}^{r} \sigma_i^2 \cancelto{1}{\sum_m u_{im}^2} \cancelto{1}{\sum_n v_{in}^2} + \sum_{i=k+1}^{r} \sum_{\substack{j=k+1 \\ j \neq i}}^{r} \sigma_i \sigma_j \cancelto{0}{\sum_m u_{im} u_{jm}} \cancelto{0}{\sum_n v_{in} v_{jn}}$$

$$= \sum_{i=k+1}^{r} \sigma_i^2 \quad \sigma_i^2 = \lambda_i$$

Remark:

- To approximate an image using SVD, arrange the eigenvalues λ_i in decreasing order and remove the last few terms in $\sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$
- rank-k approximation is the optimal approximation using k-terms
  (in term of F-norm) (or with rank-k image)

Haar transformation

(From now on, we assume all images $f = (f_{ij})_{\substack{0 \le i \le N-1 \\ 0 \le j \le N-1}}$

Definition: (Haar functions) The Haar functions are defined as follows

$$H_0(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

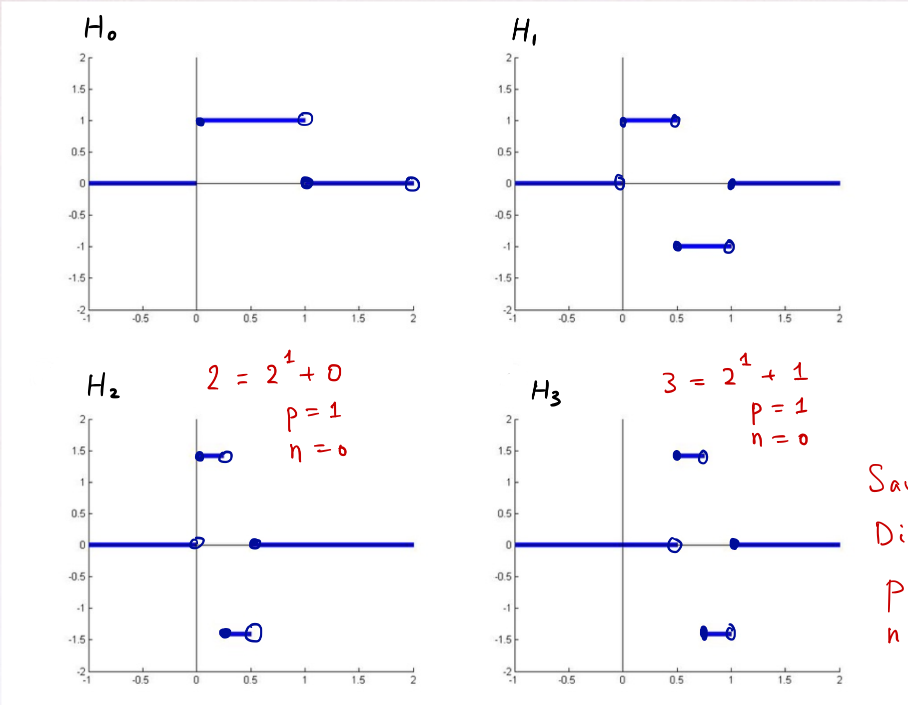
$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$H_{2^p+n} \equiv \begin{cases} \sqrt{2}^{\,p} & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^{\,p} & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p = 1, 2, \ , \ n = 0, 1, 2, \ , 2^p - 1$

Remark: If p is larger, H_{2^p+n} is compactly supported in a smaller region

Examples of Haar functions:



Same wavefront

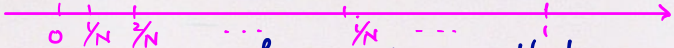
Different locations

$p \leftrightarrow$ wavefront

$n \leftrightarrow$ location

Definition (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is done by dividing $[0,1]$ into partitions



Let $H(k,i) \equiv H_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \ldots, N-1$

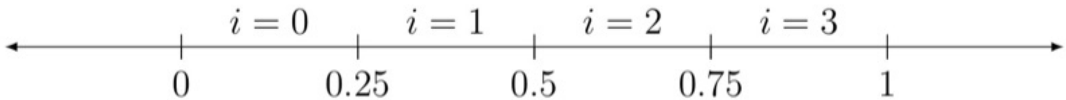
We obtain the Haar Transform matrix $\tilde{H} \equiv \frac{1}{\sqrt{N}} H$ where $H \equiv \big(H(k,i)\big)_{0 \le k, i \le N-1}$

The Haar Transform of $f \in M_{n \times n}$ is defined as

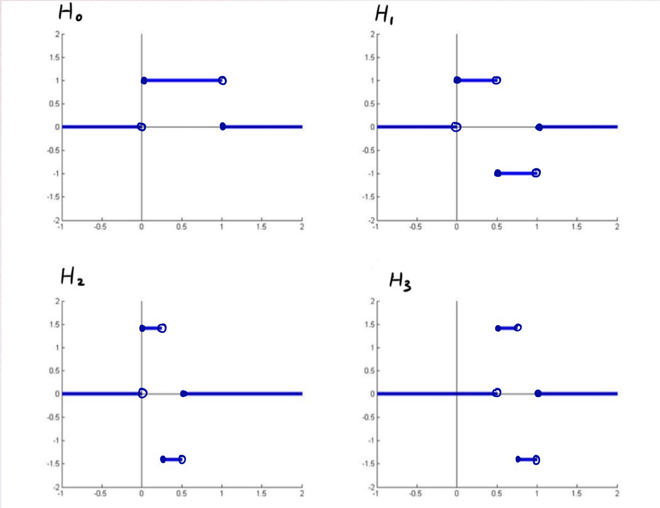
$$g = \tilde{H} f \tilde{H}^T$$

Example Compute the Haar Transform matrix for a 4×4 image.

Solution: Divide $[0,1]$ into 4 portions:



Need to check:



$$\begin{pmatrix} H(0,0) & H(0,1) & H(0,2) & H(0,3) \\ H(1,0) & H(1,1) & H(1,2) & H(1,3) \\ H(2,0) & H(2,1) & H(2,2) & H(2,3) \\ H(3,0) & H(3,1) & H(3,2) & H(3,3) \end{pmatrix}$$

$\|$

We get that:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \quad \text{and} \quad \tilde{H} = \frac{1}{\sqrt{4}} H = \frac{1}{2} H$$

Easy to check that $\tilde{H}^T \tilde{H} = I$.

$\|$

$$\begin{pmatrix} H_0(\frac{0}{4}) & H_0(\frac{1}{4}) & H_0(\frac{2}{4}) & H_0(\frac{3}{4}) \\ H_1(\frac{0}{4}) & H_1(\frac{1}{4}) & H_1(\frac{2}{4}) & H_1(\frac{3}{4}) \\ & & & \\ & & & \end{pmatrix}$$

Example 2 Compute the Haar Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

$$g = \tilde{H} f \tilde{H}^T = \left.\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\right\} \text{More zeros}$$

Example 3 Suppose g in Example 2 is changed to:

$$g = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Reconstruct the original image.

Solution:

$$f = \tilde{H}^T g \tilde{H} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Localized error

Remark:

1. Haar Transform usually produces coefficient matrix with more zeros!

2. Localized error in coefficient matrix causes localized error in the reconstructed image