Lecture 3

Recall:

Convolution

Definition: Consider $k \in M_{N\times N}(\mathbb{R})$ and $f \in M_{N\times N}(\mathbb{R})$. Assume k and f are periodically extended. That is:

$$\begin{aligned} k(x,y) &= k(x+pN,\, y+qN) \\ f(x,y) &= f(x+pN,\, y+qN) \end{aligned} \qquad \text{where } p, q \text{ are integers.}$$

The convolution $k*f$ of k and f is a N×N matrix defined as:

$$k*f(\alpha,\beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} k(x,y)\, f(\alpha-x,\, \beta-y) \quad \text{for } 1 \le \alpha, \beta \le N$$

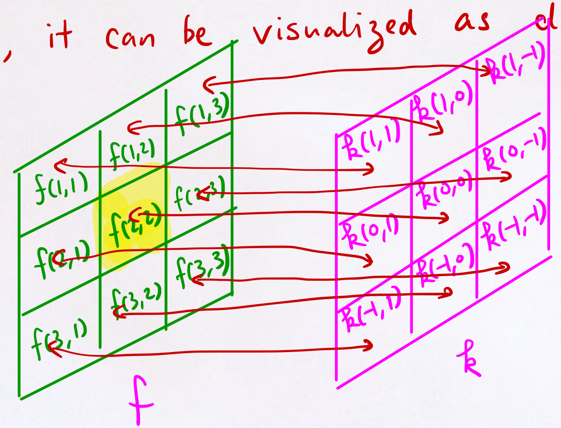
Geometric meaning of convolution

Consider $k \in M_{3\times3}(\mathbb{R})$ and $f \in M_{3\times3}(\mathbb{R})$.

Consider: $k*f(2,2) = \sum_{x=1}^{3}\sum_{y=1}^{3} k(2-x,2-y)f(x,y)$

$$= k(1,1)f(1,1) + k(1,0)f(1,2) + k(1,-1)f(1,3) + k(0,1)f(2,1) + k(0,0)f(2,2) + k(0,-1)f(2,3) + k(-1,1)f(3,1) + k(-1,0)f(3,2) + k(-1,-1)f(3,3)$$

Geometrically, it can be visualized as dot product:



Overlay k onto f and take dot product.

Definition: The point spread function $h^{\alpha,\beta}(x,y)$ of a linear image transformation is called shift-invariant if there exists a function $\tilde{h}$ such that

$$h^{\alpha,\beta}(x,y) = \tilde{h}(\alpha - x, \beta - y)$$

for all $1 \le x, y, \alpha, \beta \le N$.

Remark: Given $k \in M_{N\times N}(\mathbb{R})$. Let $\mathcal{O}$ be a linear image transformation defined by : $\mathcal{O}(f) = k * f$ for all $f \in M_{N\times N}(\mathbb{R})$.

Then : the point spread function of $\mathcal{O}$ is shift-invariant.

Let $g = \mathcal{O}(f)$

$$g(\alpha,\beta) = \mathcal{O}(f)(\alpha,\beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} \underset{\displaystyle h^{\alpha,\beta}(x,y)}{\underset{\shortparallel}{k(\alpha-x,\beta-y)}}\, f(x,y)$$

Similarity between images

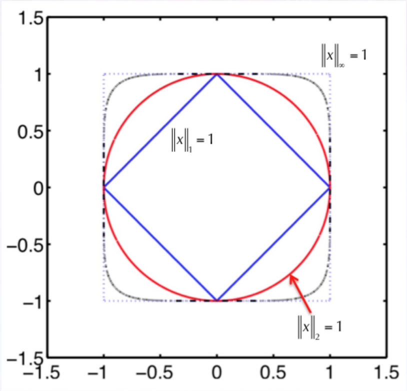
Need to define matrix norm $\|\cdot\|$ such that : for $\forall\ f, g \in \mathcal{I}$, we can define similarity between f and g as $\|f-g\|$.

Definition: A vector / matrix norm is a function $\|\cdot\| : \mathbb{R}^m$ (or $\mathbb{R}^{m\times n}$) $\to \mathbb{R}$ so that for any $\vec{x}, \vec{y} \in \mathbb{R}^m$ (or $\mathbb{R}^{m\times n}$) and $\alpha \in \mathbb{R}$, we have:

1. $\|\vec{x}\| \geq 0$, $\|\vec{x}\| = 0$ iff $\vec{x} = 0$.
2. $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ (triangle inequality)
3. $\|\alpha\vec{x}\| = |\alpha|\,\|\vec{x}\|$

Example:

$$\left.\begin{array}{l} \bullet\ \|\vec{x}\|_1 = \sum_{i=1}^{m} |x_i| \qquad \vec{x} = (x_1, x_2, \dots, x_m)^T \\ \bullet\ \|\vec{x}\|_2 = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2} \\ \bullet\ \|\vec{x}\|_\infty = \max\limits_{i=1,2,\dots,m} |x_i| \end{array}\right\} \text{Vector norm}$$



Remark: In image processing, vector norm can be considered as matrix norm.

Suppose $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & & a_{NN} \end{pmatrix} \in M_{N\times N}(\mathbb{R}) \longrightarrow \vec{A} = \begin{array}{c} \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{N1} \\ \vdots \\ a_{1N} \\ a_{2N} \\ \vdots \\ a_{NN} \end{pmatrix} \end{array} \begin{array}{l} \\ \leftarrow \text{1st col of A} \\ \\ \\ \\ \\ \leftarrow N^{th} \text{ col of A} \\ \\ \\ \end{array}$

Then: $\|A\|_1 = \|\vec{A}\|_1 = \sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}|$

Given two images A and B, similarity between then can be measured by: $\sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij} - b_{ij}|$

Another commonly used matrix norm

Definition: (Frobenius norm) $\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}$

Let $\vec{a}_j$ = j-th col of A. We have: $\|A\|_F = \sqrt{\sum_{j=1}^{n} \|a_j\|_2^2} = \sqrt{tr(A^T A)} = \sqrt{tr(AA^T)}$

where $tr(\cdot)$ = trace of the matrix.

Importance of defining correct norm

Figure 1: The images on the left and on the right are equally similar to the image in the middle in terms of the entrywise 1-norm. On the other hand, the image on the right is significant less similar to the image in the middle in terms of the entrywise 2-norm than the image on the left.

Figure 2: The images on the left and on the right are equally similar to the image in the middle in terms of the entrywise 2-norm. On the other hand, the image on the right is significant less similar to the image in the middle in terms of the entrywise 1-norm than the image on the left.

Representation of $\mathcal{O}$ by a matrix H :

Let $g = \mathcal{O}(f) \in M_{N\times N}(\mathbb{R})$.

Then: $$g(\alpha,\beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} h^{\alpha,\beta}(x,y)\, f(x,y) \quad \text{for} \quad 1 \le \alpha,\beta \le N$$

So,

$$(*)\left\{\begin{aligned}
g(1,1) &= h^{1,1}(1,1) f(1,1) + \dots + h^{1,1} f(1,N) + \dots + h^{1,1}(N,1) f(N,1) + \dots + h^{1,1}(N,N) f(N,N) \\
g(2,1) &= h^{2,1}(1,1) f(1,1) + \dots + h^{2,1} f(1,N) + \dots + h^{2,1}(N,1) f(N,1) + \dots + h^{2,1}(N,N) f(N,N) \\
&\ \vdots \\
g(\alpha,\beta) &= h^{\alpha,\beta}(1,1) f(1,1) + \dots + h^{\alpha,\beta} f(1,N) + \dots + h^{\alpha,\beta}(N,1) f(N,1) + \dots + h^{\alpha,\beta}(N,N) f(N,N) \\
&\ \vdots \\
g(N,N) &= h^{N,N}(1,1) f(1,1) + \dots + h^{N,N} f(1,N) + \dots + h^{N,N}(N,1) f(N,1) + \dots + h^{N,N}(N,N) f(N,N)
\end{aligned}\right.$$

N^2 equations, N^2 variables. LHS = $\begin{pmatrix} g(1,1) \\ g(2,1) \\ \vdots \\ g(N,1) \\ \vdots \\ g(1,N) \\ \vdots \\ g(N,N) \end{pmatrix} = \underset{\in\, \mathbb{R}^{N^2}}{\vec{g}}$

Variables on RHS: $\begin{pmatrix} f(1,1) \\ f(2,1) \\ \vdots \\ f(N,1) \\ \vdots \\ f(1,N) \\ \vdots \\ f(N,N) \end{pmatrix} = \vec{f} \in \mathbb{R}^{N^2}$

So, (*) can be written in matrix form:

$$\vec{g} = H \vec{f} \qquad \left(H \in M_{N^2 \times N^2}(\mathbb{R}) \right)$$

H is called the transformation matrix representing $\mathcal{O}$.

Example 1.1 A linear operator is such that it replaces the value of each pixel by the average of its four nearest neighbours. Assume the image is repeated in all directions. Apply this operator $\mathcal{O}$ to a 3×3 image. Find the transformation matrix corresponding to $\mathcal{O}$.

Solution:

$$3 \times 3 \text{ image} = \begin{array}{cccccccccl}
 & & & f_{11} & f_{12} & f_{13} & & & & \leftarrow \text{Row } -3 \\
 & & & f_{21} & f_{22} & f_{23} & & & & \leftarrow \text{Row } -2 \\
 & & & f_{31} & f_{32} & f_{33} & & & & \leftarrow \text{Row } -1 \\
f_{11} & f_{12} & f_{13} & f_{11} & f_{12} & f_{13} & f_{11} & f_{12} & f_{13} & \leftarrow \text{Row } 1 \\
f_{21} & f_{22} & f_{23} & f_{21} & f_{22} & f_{23} & f_{21} & f_{22} & f_{23} & \leftarrow \text{Row } 2 \\
f_{31} & f_{32} & f_{33} & f_{31} & f_{32} & f_{33} & f_{31} & f_{32} & f_{33} & \leftarrow \text{Row } 3 \\
\uparrow & \uparrow & \uparrow & f_{11} & f_{12} & f_{13} & \uparrow & \uparrow & \uparrow & \leftarrow \text{Row } 4 \\
\text{Col } -3 & \text{Col } -2 & \text{Col } -1 & f_{21} & f_{22} & f_{23} & \text{Col } 4 & \text{Col } 5 & \text{Col } 6 & \leftarrow \text{Row } 5 \\
 & & & f_{31} & f_{32} & f_{33} & & & & \leftarrow \text{Row } 6
\end{array}$$

$$g_{22} = \frac{f_{12} + f_{21} + f_{23} + f_{32}}{4} \; ; \quad g_{33} = \frac{f_{23} + f_{32} + f_{31} + f_{13}}{4}$$

etc ...

Write

$$\vec{g} = \begin{pmatrix} g_{11} \\ g_{21} \\ g_{31} \\ g_{12} \\ g_{22} \\ g_{32} \\ g_{13} \\ g_{23} \\ g_{33} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & \boxed{\frac{1}{4}} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \boxed{\frac{1}{4}} & 0 \\ & & & & \vdots & & & & \end{pmatrix} \begin{pmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{12} \\ f_{22} \\ f_{32} \\ f_{13} \\ f_{23} \\ f_{33} \end{pmatrix}$$

$h^{1,1}(3,1)$

$h^{2,1}(2,3)$

etc

$$g_{11} = \frac{f_{13} + f_{31} + f_{12} + f_{21}}{4}$$

$$g_{21} = \frac{f_{11} + f_{22} + f_{31} + f_{23}}{4}$$

By careful examination, we see that

$$\begin{bmatrix} 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 0 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 1/4 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 & 0 & 0 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 0 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 0 & 0 & 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 \end{bmatrix}$$

All entries are given by the point spread function

$h^{\alpha,\beta}(x,y)$

Example Consider $\Phi : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ defined by:

$$\Phi(f) = \underset{\overset{\shortparallel}{A}}{\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}} f \underset{\overset{\shortparallel}{B}}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \text{ for all } f \in M_{N\times N}(\mathbb{R}).$$

Let $g = \Phi(f)$. Then:

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_{11} + 3f_{12} & 2f_{11} + 4f_{12} \\ f_{11} + 2f_{21} + 3f_{12} + 6f_{22} & 2f_{11} + 4f_{21} \\ & + 4f_{12} + 8f_{22} \end{pmatrix}$$

$$\begin{pmatrix} g_{11} \\ g_{21} \\ g_{12} \\ g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 1 & 2 & 3 & 6 \\ 2 & 0 & 4 & 0 \\ 2 & 4 & 4 & 8 \end{pmatrix} \begin{pmatrix} f_{11} \\ f_{21} \\ f_{12} \\ f_{22} \end{pmatrix}$$

$$\overset{\shortparallel}{\begin{pmatrix} 1\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} & 3\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \\ 2\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} & 4\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \end{pmatrix}} = \begin{pmatrix} 1A & 3A \\ 2A & 4A \end{pmatrix}$$

Remark: • Separable image transformation has a special structure.

- Let A and B be two matrices.

Kronecker product of A and B $=$

$$\underset{(a_{ij})_{1\le i,j\le N}}{\underset{\parallel}{A}} \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1N}B \\ a_{21}B & \cdots & a_{2N}B \\ \vdots & \vdots & \\ a_{N1}B & \cdots & a_{NN}B \end{pmatrix}$$

- In general, if $\mathcal{O}: M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ is defined by:

$$\mathcal{O}(f) = AfB \text{ for all } f \in M_{N\times N}(\mathbb{R}), \text{ where } A, B \in M_{N\times N}(\mathbb{R})$$

Then, the transformation matrix of $\mathcal{O}$ is:

$$H = B^T \otimes A$$

- So, instead of storing $N^2 \times N^2 = N^4$ entries, we only need to store entries of A and B, which is $2N^2$ (much less storage)

Image decomposition

Let $g = A f B$ (Separable). Then: $f = A^{-1} g B^{-1}$

(assume that A and B are invertible)

Write: $A^{-1} = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}$; $B^{-1} = \begin{pmatrix} — & \vec{v}_1^T & — \\ — & \vec{v}_2^T & — \\ & \vdots & \\ — & \vec{v}_N^T & — \end{pmatrix}$

Then: $$g = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{\vec{u}_i \vec{v}_j^T}_{M_{N \times N}}$$

Proof: $$f = A^{-1} f B^{-1} = A^{-1} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} f_{ij} \begin{array}{c} \quad j^{th}\text{ col} \\ \quad \downarrow \\ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & 1 & 0 \\ 0 & \cdots & 0 \end{pmatrix} \end{array} \leftarrow i^{th}\text{ row} \right) B^{-1}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} f_{ij} \underbrace{A^{-1} \begin{pmatrix} 0 & \cdot & 0 & & 0 \\ \vdots & & \vdots & & \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \\ 0 & & 0 & \cdots & 0 \end{pmatrix} B^{-1}}_{} = \begin{pmatrix} | \\ \vec{u}_i \\ | \end{pmatrix} \begin{pmatrix} — \vec{v}_j^T — \end{pmatrix}$$

$\| \quad N \times N$ matrix

$\therefore f =$ linear combination of $\{\vec{u}_i \vec{v}_j^T\}_{i,j}$

Definition: Each $\vec{u}_i v_j^T$ is called an elementary image.

$\vec{u}_i v_j^T$ is also called the outer product of $\vec{u}_i$ and $\vec{v}_j$.

One important task in image processing:

Choose A and B such that:

1. Transformed image requires less storage (Many $g_{ij} = 0$)
2. Take away some terms $g_{ij} \vec{u}_i \vec{v}_j^T$ (e.g. high-frequency) → Better image!!
3. A^{-1} and B^{-1} are easy to compute!

Common example:

Orthogonal $U \Leftrightarrow U^T U = I \quad \therefore U^{-1} = U^T$.

Example: Let $A = \underbrace{\begin{pmatrix}1&2\\1&1\end{pmatrix}}_{A^{-1}}\underbrace{\begin{pmatrix}1&0\\0&2\end{pmatrix}}_{g}\underbrace{\begin{pmatrix}3&1\\1&3\end{pmatrix}}_{B^{-1}}$

Then: $$A = \begin{pmatrix}1&2\\1&1\end{pmatrix}\left[1\begin{pmatrix}1&0\\0&0\end{pmatrix}+2\begin{pmatrix}0&0\\0&1\end{pmatrix}\right]\begin{pmatrix}3&1\\1&3\end{pmatrix}$$

$$= 1\begin{pmatrix}1&2\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}3&1\\1&3\end{pmatrix} + 2\begin{pmatrix}1&2\\1&1\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}3&1\\1&3\end{pmatrix}$$

$$= \underset{\substack{\shortparallel\\ g_{11}}}{1}\begin{pmatrix}1&2\\1&1\end{pmatrix}\begin{pmatrix}3&1\\0&0\end{pmatrix} + \underset{\substack{\shortparallel\\ g_{22}}}{2}\begin{pmatrix}1&2\\1&1\end{pmatrix}\begin{pmatrix}0&0\\1&3\end{pmatrix}$$

$$\begin{matrix}\shortparallel\\ \underset{\vec{u}_1}{\begin{pmatrix}1\\1\end{pmatrix}}\underset{\vec{v}_1^{\,T}}{(3,1)}\end{matrix} \qquad\qquad \begin{matrix}\shortparallel\\ \underbrace{\begin{pmatrix}2\\1\end{pmatrix}}_{\vec{u}_2}\underbrace{(1\ \ 3)}_{\vec{v}_2^{\,T}}\end{matrix}$$

$$= 1\underbrace{\begin{pmatrix}3&1\\3&1\end{pmatrix}}_{\text{elementary image}} + 2\underbrace{\begin{pmatrix}2&6\\1&3\end{pmatrix}}_{\text{elementary image}}$$

Image decomposition

Image decomposition based on Singular Value Decomposition (SVD)

Definition: (SVD) For any $g \in M_{m \times n}$, the singular value decomposition (SVD) of g is a matrix factorization: $g = U \Sigma V^T$, where $U \in M_{m \times m}$, $V \in M_{n \times n}$ are orthogonal, $\Sigma \in M_{m \times n}$ is a diagonal matrix ($\Sigma_{ij} = 0$ if $i \neq j$) with diagonal entries given by:

$$\underbrace{\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r}_{\text{singular values}} > 0 \quad \text{with } r \leq \min(m, n). \qquad (UU^T = U^T U = I \; ; \quad VV^T = V^T V = I)$$

Theorem: The rank of g is given by the number of non-zero singular values.

Proof: Rank = dim of column space.

Recall that $\text{rank}(AB) = \text{rank}(B)$ if A is invertible

$\text{rank}(AB) = \text{rank}(A)$ if B is invertible.

Suppose $g = U \Sigma V^T$. Since U and V are invertible, $\text{rank}(g) = \text{rank}(\Sigma)$

$= \#$ of non-zero singular values

Remark: Consider an image g. Let $g = U\Sigma V^T$ be the SVD of g (with diagonal entries of Σ given by $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$)

1. Note that
$$g = U\Sigma V^T = \sum_{i=1}^{r} \sigma_i \, U \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & \overset{\substack{\text{ith}\\ \downarrow}}{1} & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \leftarrow \text{ith} \; V^T = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

$\vec{u}_i \vec{v}_i^T$ is called the eigen-image of g under SVD.

2. For $N \times N$ image, the required storage is:
$$(\underbrace{N}_{\vec{u}_i} + \underbrace{N}_{\vec{v}_i} + \underbrace{1}_{\sigma_i}) \times \underbrace{r}_{r\text{ terms}} = (2N+1)r$$

Observation about SVD

Let $A = U \Sigma V^T$ (Let $A \in M_{n \times n}$)

Write $U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{pmatrix}$; $V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}$; $\Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix}$

- $A^T A = (U \Sigma V^T)^T (U \Sigma V^T) = V \Sigma^T \underbrace{U^T U}_{I} \Sigma V^T = V \Sigma^T \Sigma V^T$

$$\Rightarrow (A^T A) V = V \begin{pmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{pmatrix} \Rightarrow \begin{pmatrix} | & & | \\ (A^T A)\vec{v}_1 & \cdots & (A^T A)\vec{v}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \sigma_1^2 \vec{v}_1 & \cdots & \sigma_n^2 \vec{v}_n \\ | & & | \end{pmatrix}$$

$$\Rightarrow (A^T A)\vec{v}_1 = \sigma_1^2 \vec{v}_1 , \ldots , (A^T A)\vec{v}_n = \sigma_n^2 \vec{v}_n$$

$\therefore$ $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are eigenvectors of $A^T A$ with eigenvalues $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$.

- $AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma \underbrace{V^T V}_{I} \Sigma^T U^T = U\Sigma\Sigma^T U^T$

$$\Rightarrow (AA^T)U = U\begin{pmatrix}\sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2\end{pmatrix} \Rightarrow \begin{pmatrix} | & & | \\ AA^T\vec{u}_1 & \cdots & AA^T\vec{u}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \sigma_1^2\vec{u}_1 & \cdots & \sigma_n^2\vec{u}_n \\ | & & | \end{pmatrix}$$

$\therefore$ $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ are eigenvectors of AA^T with eigenvalues $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$

- $A = U\Sigma V^T \Rightarrow AV = U\Sigma \Rightarrow \begin{pmatrix} | & | & & | \\ A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \sigma_1\vec{u}_1 & \sigma_2\vec{u}_2 & \cdots & \sigma_n\vec{u}_n \\ | & | & & | \end{pmatrix}$

$\therefore$ For $\sigma_1, \sigma_2, \dots, \sigma_r > 0$,

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1}, \quad \vec{u}_2 = \frac{A\vec{v}_2}{\sigma_2}, \quad \dots, \quad \vec{u}_r = \frac{A\vec{v}_r}{\sigma_r}$$

We can obtain $\vec{u}_1, \dots, \vec{u}_r$ from $\vec{v}_1, \dots, \vec{v}_r$.

- $U^T U = I \Rightarrow \vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

  $V^T V = I \Rightarrow \vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

$\therefore \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ are orthonormal.

$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are orthonormal.

How to compute SVD

Let $A \in M_{n\times n}$

Step 1: Find eigenvalues $\{\overset{\sigma_1^2}{\overset{\shortparallel}{\lambda_1}}, \overset{\sigma_2^2}{\overset{\shortparallel}{\lambda_2}}, \ldots, \overset{\sigma_n^2}{\overset{\shortparallel}{\lambda_n}}\}$
and orthonormal eigenvectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$
of $A^TA \in M_{n\times n}$ (with $\|\vec{v}_j\| = 1$, $j=1,\ldots,n$)

[Recall: $(A^TA)\vec{v}_j = \lambda_j \vec{v}_j$]

Step 2: Define:

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix} \in M_{n\times n}$$

Step 3: For non-zero $\sigma_1, \sigma_2, \ldots, \sigma_r$,

let $\vec{u}_1 = \dfrac{A\vec{v}_1}{\sigma_1}$, $\vec{u}_2 = \dfrac{A\vec{v}_2}{\sigma_2}$, $\ldots$, $\vec{u}_r = \dfrac{A\vec{v}_r}{\sigma_r}$

Step 4: Extend $\{\vec{u}_1, \ldots, \vec{u}_r\}$ to the basis
$\{\vec{u}_1, \ldots, \vec{u}_r, \ldots, \vec{u}_n\}$ of $\mathbb{R}^n$

Step 5: Let:

$$U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \\ | & | & & | \end{pmatrix} \in M_{m\times m}$$

$$V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{n\times n}$$

Then: $A = U \Sigma V^T$

Example: (2×2 example) Find the SVD of $A = \begin{pmatrix} 4 & 0 \\ 3 & -5 \end{pmatrix}$

Step 1: $A^TA = \begin{pmatrix} 25 & -15 \\ -15 & 25 \end{pmatrix}$

Characteristic polynomial: $\det(A^TA - \lambda I) = (25-\lambda)(25-\lambda) - 15^2$

$$= \lambda^2 - 50\lambda + 400$$

$$= (\lambda - 10)(\lambda - 40)$$

∴ A^TA has two eigenvalues: $\lambda = 10$ and $\lambda = 40$

For $\lambda = 40$, $\vec{0} = (A^TA - 40I) = \begin{pmatrix} -15 & -15 \\ -15 & -15 \end{pmatrix} \overset{RREF}{\rightsquigarrow} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. $\vec{v} = x_1 \begin{pmatrix} -1 \\ +1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Choose $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} / \sqrt{2} = \begin{pmatrix} -1/\sqrt{2} \\ +1/\sqrt{2} \end{pmatrix}$

For $\lambda = 10$, $(A^TA - 10I) = \begin{pmatrix} 15 & -15 \\ -15 & 15 \end{pmatrix}$

Find null space to find eigenvector.

RREF of $\begin{pmatrix} 15 & -15 \\ -15 & 15 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = 0$

Let $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ = eigenvecto

Then: $x_1 - x_2 = 0$

∴ $\vec{v} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Choose $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \sqrt{1^2 + 1^2} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$$\therefore \quad V = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{pmatrix}$$

Step 2:

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \begin{pmatrix} -1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix} \quad \text{and} \quad \vec{u}_2 = \frac{A\vec{v}_2}{\sigma_2} = \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}.$$

$\therefore$ SVD of A is:

$$\begin{pmatrix} 4 & 0 \\ 3 & -5 \end{pmatrix} = \underbrace{\begin{pmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{V^T}$$