Lecture 2:

Recap:

Definition: (Linear image transformation)

An image transformation $\mathcal{O}: M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ is linear if it satisfies:

$\mathcal{O}(af + bg) = a\,\mathcal{O}(f) + b\,\mathcal{O}(g)$ for all $f, g \in M_{N\times N}(\mathbb{R})$, $a, b \in \mathbb{R}$.

Point Spread Function

Take $f \in \mathcal{I} = M_{N \times N}(\mathbb{R})$.

$$\text{Let } f = \begin{pmatrix} f(1,1) & \cdots & f(1,N) \\ f(2,1) & & f(2,N) \\ \vdots & f(x,y) & \vdots \\ f(N,1) & \cdots & f(N,N) \end{pmatrix} = \sum_{x=1}^{N} \sum_{y=1}^{N} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & f(x,y) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}. = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y) \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

(x, y) entry

Consider a linear image transformation $\mathcal{O} : M_{N \times N}(\mathbb{R}) \to M_{N \times N}(\mathbb{R})$.

Let $g = \mathcal{O}(f)$. Then:

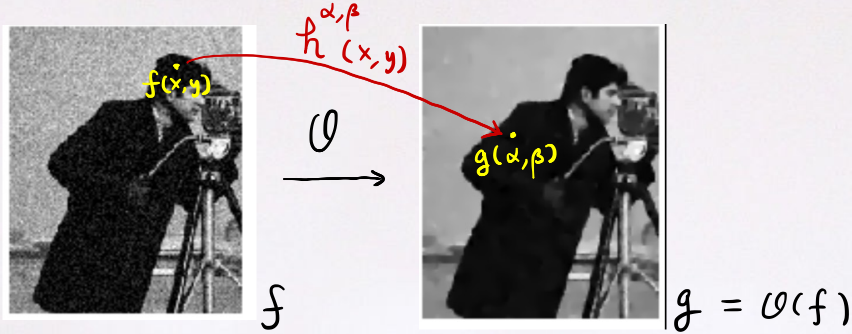
$$g(\alpha, \beta) = \left[\sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, \mathcal{O}\left(\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \right) \right]_{\alpha, \beta}$$

$$= \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, h^{\alpha, \beta}(x,y)$$

where

$$h^{\alpha,\beta}(x,y) = \left[\mathcal{O}(P_{xy}) \right]_{\alpha,\beta} \; ; \; P_{xy} = \begin{array}{cc} \begin{array}{ccccc} & & y^{th} & & \\ & & \downarrow & & \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} & \begin{array}{c} \\ \\ \leftarrow x^{th} \\ \\ \\ \end{array} \end{array}$$

Remark: $h^{\alpha,\beta}(x,y)$ determines how much the pixel value of f at (x,y) influences the pixel value of g at (α,β).



Definition: (Point spread function)

$h^{\alpha,\beta}(x,y)$ is usually called the point spread function (PSF)

Separable linear image transformation

Definition: An image transformation $\mathcal{O} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ is said to be separable if there exists matrices $A \in M_{N\times N}(\mathbb{R})$ and $B \in M_{N\times N}(\mathbb{R})$ such that: $\mathcal{O}(f) = AfB$ for all $f \in M_{N\times N}(\mathbb{R})$.

Theorem: Let $\mathcal{O}$ be a separable image transformation given by: $\mathcal{O}(f) = AfB$ for all $f \in M_{N\times N}(\mathbb{R})$, where $A, B \in M_{N\times N}(\mathbb{R})$. Then, the point spread function of $\mathcal{O}$ is given by:

$$h^{\alpha,\beta}(x,y) = A(\alpha,x)\, B(y,\beta)$$

where $A(\alpha,x)$ is the (α,x) entry of A, $B(y,\beta)$ is the (y,β) entry of B.

Periodic extension of images

Let $f \in M_{N\times N}(\mathbb{R})$ be an image. We say f is periodically extended

if $f(x,y) = f(x+pN, y+qN)$ where p, q are integers.

(So, f can now be defined on the entire plane $-\infty < x, y < \infty$)

For example, $f(N+10, 2N+5) = f(10, 5)$

Geometrically, suppose $f \in M_{3\times3}(\mathbb{R})$

$$\begin{array}{cccl}
 & & \downarrow \text{col } 5 & \\
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} & \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} & \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} & \begin{array}{l} \leftarrow \text{row } -2 \\ \leftarrow \text{row } -1 \\ \leftarrow \text{row } 0 \end{array} \\
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} & \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} & \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} & \begin{array}{l} \leftarrow \text{row } 1 \\ \\ \\ \end{array} \\
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} & \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} & \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\end{array}$$

$\therefore f(1, 5) = b$

$f(-2, 0) = c$

Convolution

Definition: Consider $k \in M_{N\times N}(\mathbb{R})$ and $f \in M_{N\times N}(\mathbb{R})$. Assume k and f are periodically extended. That is:

$$k(x,y) = k(x+pN, y+qN)$$
$$f(x,y) = f(x+pN, y+qN)$$

where p, q are integers.

The convolution $k * f$ of k and f is a $N\times N$ matrix defined as:

$$k*f(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} k(x,y)\, f(\alpha - x, \beta - y) \quad \text{for } 1 \le \alpha, \beta \le N$$

Example: Let $k = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $f = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Find $k * f \in M_{2\times 2}(\mathbb{R})$.

The $(1,1)$ entry of $k * f$ is defined as:

$$k * f(1,1) = \sum_{x=1}^{2} \sum_{y=1}^{2} k(x,y)\, f(1-x,\, 1-y)$$

$$= k(1,1)\underbrace{f(0,0)}_{= f(2,2)} + k(1,2)\underbrace{f(0,-1)}_{= f(2,1)} + k(2,1)\underbrace{f(-1,0)}_{= f(1,2)} + k(2,2)\underbrace{f(-1,-1)}_{= f(1,1)}$$

$$= (1)(1) + (2)(1) + (3)(2) + (4)(1) = 13.$$

Similarly, $k * f(1,2) = 14$, $k * f(2,1) = 11$, $k * f(2,2) = 12$.

$$\therefore \quad k * f = \begin{pmatrix} 13 & 14 \\ 11 & 12 \end{pmatrix}.$$

$$f(3,2) = f(0,-1)$$

$$f = \begin{array}{cccc}
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{array}{l} \leftarrow \text{Row } -2 \\ \leftarrow \text{Row } -1 \\ \leftarrow \text{Row } 0 \end{array} \\
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{array}{l} \leftarrow \text{Row } 1 \\ \leftarrow \text{Row } 2 \\ \leftarrow \text{Row } 3 \end{array} \\
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &
\begin{array}{l} \leftarrow \text{Row } 4 \\ \leftarrow \text{Row } 5 \\ \leftarrow \text{Row } 6 \end{array} \\
\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{Col } -2 & \text{Col } -1 & \text{Col } 0 \end{array} &
\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{Col } 1 & \text{Col } 2 & \text{Col } 3 \end{array} &
\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{Col } 4 & \text{Col } 5 & \text{Col } 6 \end{array} &
\end{array}$$

Example: Let $k = \begin{pmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{pmatrix}$ and $f \in M_{3\times 3}(\mathbb{R})$. Find $k * f(2,2)$

$$k * f(2,2) = \sum_{x=1}^{3} \sum_{y=1}^{3} k(x,y)\, f(2-x, 2-y)$$

$$= \frac{1}{9} f(1,1) + \frac{1}{9} f(1,0) + \frac{1}{9} f(1,-1) + \frac{1}{9} f(0,1) + \frac{1}{9} f(0,0) + \frac{1}{9} f(0,-1)$$

$$+ \frac{1}{9} f(-1,1) + \frac{1}{9} f(-1,0) + \frac{1}{9} f(-1,-1)$$

$$= \frac{f(1,1) + f(1,2) + f(1,3) + f(2,1) + f(2,2) + f(2,3) + f(3,1) + f(3,2) + f(3,3)}{9}$$

(Averaging the intensity values in the neighborhood of $f(2,2)$)

Remark: Averaging is commonly used in image processing, which is related to convolution.

Theorem: Let $k \in M_{N\times N}(\mathbb{R})$. Define $\mathcal{U} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ by:

$$\mathcal{U}(f) = k * f \quad \text{for all} \quad f \in M_{N\times N}(\mathbb{R}).$$

Then: $\mathcal{U}$ is linear.

Pf: Followed from the definition of convolution.

Theorem: Let $k \in M_{N\times N}(\mathbb{R})$ and $f \in M_{N\times N}(\mathbb{R})$. Then: $k * f = f * k$.

Proof: Assuming $\alpha, \beta > 1$,

$$k * f(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} k(x,y)\, f(\alpha - x, \beta - y)$$

$$= \sum_{\tilde{x}=\alpha-N}^{\alpha-1} \sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \tilde{y}) \quad (\text{let } \tilde{x} = \alpha - x,\ \tilde{y} = \beta - y)$$

$$\overset{??}{=} \sum_{\tilde{x}=1}^{N} \sum_{\tilde{y}=1}^{N} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \tilde{y}) = f * k(\alpha, \beta).$$

$$\sum_{\tilde{x}=\alpha-N}^{\alpha-1} \sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \hat{y})$$

$$= \sum_{\tilde{x}=\alpha-N}^{0} \left(\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \tilde{y}) \right) + \sum_{\tilde{x}=1}^{\alpha-1} \left(\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \hat{y}) \right)$$

$$k(\alpha-\tilde{x}, \beta-\tilde{y}) = k(\alpha-(\tilde{x}+N), \beta-\tilde{y}) \qquad f(\tilde{x}, \tilde{y}) = f(\tilde{x}+N, \tilde{y})$$

(periodic extension)

$$= \sum_{\tilde{x}=\alpha}^{N} \left(\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \hat{y}) \right) + \sum_{\tilde{x}=1}^{\alpha-1} \left(\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \hat{y}) \right)$$

$$= \sum_{\tilde{x}=1}^{N} \left(\sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \hat{y}) \right) = \sum_{\tilde{x}=1}^{N} \sum_{\tilde{y}=1}^{N} k(\alpha-\tilde{x}, \beta-\tilde{y})\, f(\tilde{x}, \tilde{y})$$

$$= f * k(\alpha, \beta)$$

The case when $\alpha = 1$ or $\beta = 1$ can be shown similarly.

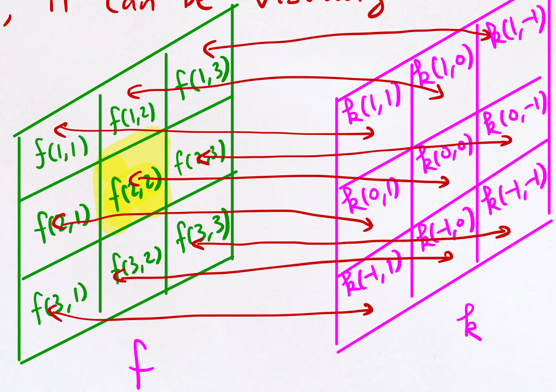
Geometric meaning of convolution

Consider $k \in M_{3\times 3}(\mathbb{R})$ and $f \in M_{3\times 3}(\mathbb{R})$.

Consider: $$k * f(2,2) = \sum_{x=1}^{3} \sum_{y=1}^{3} k(2-x, 2-y) f(x,y)$$

$$= k(1,1)f(1,1) + k(1,0)f(1,2) + k(1,-1)f(1,3) + k(0,1)f(2,1) + k(0,0)f(2,2)$$
$$+ k(0,-1)f(2,3) + k(-1,1)f(3,1) + k(-1,0)f(3,2) + k(-1,-1)f(3,3)$$

Geometrically, it can be visualized as dot product:



Overlay k onto f and take dot product.

Example: Let $\mathcal{O} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ be a linear image transformation defined by: $\mathcal{O}(f)(\alpha,\beta) = f(\alpha+1,\beta) + 2f(\alpha,\beta) - 2f(\alpha-1,\beta) + f(\alpha,\beta+1) - 2f(\alpha,\beta-1)$ for all $1 \le \alpha,\beta \le N$ and $f \in M_{N\times N}(\mathbb{R})$. Show that $\mathcal{O}$ can be expressed in terms of a convolution.

Suppose $\mathcal{O}(f) = k * f$ for some $k \in M_{N\times N}(\mathbb{R})$. Then,

Then: $\mathcal{O}(f)(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} k(\alpha-x,\beta-y)\, f(x,y)$

$$= \dots + \overset{1}{\overset{\shortparallel}{k(-1,0)}}\, f(\alpha+1,\beta) + \overset{2}{\overset{\shortparallel}{k(0,0)}}\, f(\alpha,\beta) + \overset{-2}{\overset{\shortparallel}{k(1,0)}}\, f(\alpha-1,\beta) + \overset{1}{\overset{\shortparallel}{k(0,-1)}}\, f(\alpha,\beta+1)$$

$$+ \underset{-2}{\underset{\shortparallel}{k(0,1)}}\, f(\alpha,\beta-1) + \dots$$

We set $k(-1,0) = k(N-1,N) = 1$, $k(0,0) = k(N,N) = 2$, $k(1,0) = k(1,N) = -2$
$k(0,-1) = k(N,N-1) = 1$, $k(0,1) = k(N,1) = -2$ and $k(x,y) = 0$ otherwise.

Then: $\mathcal{O}(f) = k * f$.

Definition: The point spread function $h^{\alpha,\beta}(x,y)$ of a linear image transformation is called shift-invariant if there exists a function $\tilde{h}$ such that

$$h^{\alpha,\beta}(x,y) = \tilde{h}(\alpha - x, \beta - y)$$

for all $1 \le x, y, \alpha, \beta \le N$.

Remark: Given $k \in M_{N\times N}(\mathbb{R})$. Let $\mathcal{O}$ be a linear image transformation defined by: $\mathcal{O}(f) = k * f$ for all $f \in M_{N\times N}(\mathbb{R})$.

Then: the point spread function of $\mathcal{O}$ is shift-invariant.

Let $g = \mathcal{O}(f)$

$$g(\alpha,\beta) = \mathcal{O}(f)(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} \underbrace{k(\alpha - x, \beta - y)}_{= h^{\alpha,\beta}(x,y)} f(x,y)$$